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PHILOSOPHICAL LOGIC IN POLAND

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PREFACE

The label "philosophical logic" traditionally arouses some reservations among Polish logicians. Perhaps they remember Łukasiewicz's statement made in his influential *Elements of Mathematical Logic* (published in 1929) that philosophical logic is a mixture of logic, epistemology and psychology which should be abandoned in favour of mathematical logic. Even some contributors to the present volume expressed their reservations toward its title. However, although I am fully aware that the meaning of the term "philosophical logic" is not univocally determined, I decided to use it as a signal of the content of the papers included below. I did this for at least two reasons. First, although the rubric "philosophical logic" is not completely delimited, it very conveniently covers a variety of logical studies which are of interest for philosophers. Second, as I see the matter, this label expresses the evaluative attitude that logic has utmost relevance for philophers. This conviction was a very important ingredient of the ideology of the Polish Logical School in the interwar period (including Łukasiewicz himself) and it is commonly shared by the present Polish logical community. This collection gives a strong evidence for this.

The papers included in this volume vary in their subjects. Two papers by Father Bocheński (the doyen of Polish logicians; metaphorically speaking, a bridge between old and new times) touch on general problems. The opening paper is not only a personal report but also a documentation of how duties of logic were understood in the Golden Age of "Polish logic". The second paper of Bocheński outlines his idea of how logic and philosophy are mutually related. Then we have studies more or less devoted to formal logical matters (the papers by Malinowski, Murawski, Omyła, Paśniczek, Pogorzelski, Skura, and Trzęsicki). Several contributions have topics inspired by computer science (the papers by Marciszewski, Orłowska, and Rauszer). Applications of logic to ontology are given by Jadacki, Nieznański, Suszko, Turnau, and Wojciechowski. Epistemological questions are considered by Grzegorczyk, Hiż, Placek, Wolniewicz, Woleński, and Wójcicki.

Jan Woleński (ed.), Philosophical Logic in Poland, vii–viii.

In another perspective, the topic considered vary from non-standard methods of codification of classical logic to the philosophy of science. However, this thematic multiplicity reflects the present situation in philosophical logic in every region in which it is cultivated.

I am indebted to Jaakko Hintikka for his inspiration for editing a volume presenting Polish works in philosophical logic. Peter Simons, a great friend of "Polish logic", read most papers and suggested an innumerable number of corrections. Some linguistic corrections were suggested by Arthur Szylewicz. All papers collected in this volume appear here in English for the first time.

Jan Woleński

JÓZEF M. BOCHEŃSKI

MORALS OF THOUGHT AND SPEECH – REMINISCENCES

The undersigned is sometimes said to be one of the last surviving members of the Lwów-Warsaw school. This is not quite correct, as I did not study philosophy or logic under any of its masters. Yet I knew quite a number among them and they deeply influenced my thought. Thus it will perhaps not be useless, if I write down some reminiscences of four of them.

Three thinkers are usually mentioned when one speaks about logic in Poland between the Wars: Leśniewski, Łukasiewicz and Tarski. In fact, they may be considered as the most important logicians of the school. Leśniewski and Łukasiewicz belong to the older generation and were pupils of Twardowski, while Tarski was the pupil of both. Alongside them a number of other sometimes prominent logicians lied between the two Wars in Poland. Warsaw, in particular, had numerous creative logicians. I am going to say a few words about one of them, Sobociński.

The following may give an idea of what Warsaw was as a centre of logical studies at that time. Somewhere around 1935 in my research I had some difficulties with the theory of types and wrote to Sobociński (who was at that time I think Leśniewski's assistant) asking him for help. He answered me saying: "Which one? Because we have here, you know, at least a dozen different theories of types and I think that at least three of them would not cause the difficulty you are writing about." I believe that there were not many other centres in the world of which one could say the same.

But Sobociński belonged to a younger generation. Let us therefore postpone things I have to say about him and start with some reminiscences concerning his masters.

I knew Łukasiewicz pretty well. He was very kind to me, so much so that I may speak of something like a friendship between us. Whenever in Warsaw, I was a frequent guest in his house. Łukasiewicz was a rather small, neatly dressed and rather shy person. He was not as distracted as another great logician I knew, Paul Bernays (whom I had to lead once to

Jan Woleński (ed.), Philosophical Logic in Poland, 1–8.

his own apartment because he could not find it) – but he belonged still to the same type of scholar who concentrated on their scientific business, that the world did not seem to exist for them.

Here is one story about him. I cam once to see him after supper. He received me sitting at his typewriter on which he was typing one of his interminable formulae (as every logician knows, you can type a Łukasiewiczian formula on an ordinary typewriter). Seeing me, he pulled the sheet on which he was writing from the machine and gave it to me, saying: "Do you see how beautiful and how obviously true?!" Now the formulae began with something like

$$CCCCKNCCCAKACCCKKKKNNAACA\ldots$$

I must confess that the only thing which was obvious to me was that the term "obvious" is a very relative one: for that "$CCCCK\ldots$" was perhaps obvious to him, but certainly not to me.

What struck me most at that time, however, was not so much the "obvious" as the "beautiful". And I still think that it was most characteristic of him, because for Łukasiewicz science – a scientific sentence, proof, theory, and so on – should not only be true, but also beautiful. He wrote once that a scientific paper should be also an aesthetic achievement – written in perfect language, beautiful, so precise, that not a single world could be added to or deleted from it. And I would say that his own writings are a model in that respect: they are really beautiful.

Now, many a writer does feel like him about the necessity of writing nicely. But what distinguishes Łukasiewicz from them is that he combined that love of beauty with an equally intense love of rigour. In fact, I have never met anybody who would practise a more rigorous way of writing and thinking than he did. In those old times we were practically all addicted to the ways of the *Principia Mathematica*, i.e. to the methods used in common mathematics. And even now I rarely read papers which are up to the Łukasiewiczian standards of rigour.

To mention just one aspect of that rigour, his algorithms are composed of two parts. When the operations indicated in them are performed, they turn into perfectly homomorphic formulae, so much so that in order to test the correctness of the proof it is enough to superpose one part over the other and look through them towards a light. The versification consists here in observing the contact of two bodies – something towards which all human science tends and which remains often just an ideal.

I cannot resist the temptation to illustrate this with a simple proof in Łukasiewiczian fashion:

Axioms

(1) CCpqCCrpCrq.
(2) CAppp
(3) CpApq

Rules

a. *Rule of substitution of variables* (with a recursive definition of wffs.; application symbolized by "/")
b. *Rule of detachment* (applications symbolized by "*C*")

Algorithm (Łukasiewicz called it "derivational line")

$1p/App, q/p, r/p \times C2 - C3q/p - 4$

Theorem

(4) Cpp

The algorithm reads "Take axiom No. 1. Put "*App*" for "*p*", "*p*" for "*q*" and "*p*" for "*r*". You will obtain a formula composed of (1) the letter "*C*", (2) the axiom No. 2, (3) the letter "*C*", (4) the axiom No. 3 in which "*g*" has been substituted by "*p*", (5) the theorem No. 4.

The axiom No. 1 runs:

a.	$CC\,p\,q\,CCr\,p\,C\,rq$	we substitute in it
	App for p, p for q and p for r	and obtain
b.	$CCApp\,pCCpAppCpp$	

The right-hand side of the algorithm has

	$C2 - C3\,q/p - 4$	we replace the figures with the
c.	$CAppCpApp\,Cpp$	formulae designed by them – perform the substitution indicated (q/p)

and obtain

$CApppCCpAppCpp$

i.e. a formula of exactly the same shape as that obtained by executing the left-hand side of the algorithm.

With that ideal of rigour proposed and practised by him, one cannot

wonder that Łukasiewicz used to say "Because of lack of space we cannot offer a rigorous proof and shall apply the usual method of the mathematicians" – which according to him "proceed like hares – by leaps of intuition".

The learned readers of this book will surely know that a similar union of love for rigour and beauty was not quite new with Łukasiewicz. It was already present in some Neo-Platonic thinkers, like Alexander of Alexandria. Łukasiewicz shared that attitude with another important logician of my times, Heinrich Scholz; this explains the friendship that existed between them.

And the similarity of Łukasiewicz's (and Scholz's) thinking with that of the Neo-Platonicists goes further than that. In 1936, at a meeting of what we called "The Cracow Circle" he said that whenever he works on a logical problem, he has the feeling of standing in the face of a powerful structure, harder than iron and concrete, in which he cannot change anything, and just discovers its details. Scholz told me once, by the way, that he believes the negation "must be somewhere". They were sturdy Platonists both of them.

One would think that a man like him, exclusively devoted to the study of abstract entities, a lover of formal thinking and of beauty, would not be a practical person. As a matter of fact, it seems to be that he blundered badly in his private affairs. But, when in the reconstruction of Poland he was called to assume the duties of the Minister of Public Instruction, he is said to have managed his department with success. That was said at least: knowing him, I still have some doubts concerning his practical abilities.

That Łukasiewicz was a Platonist does not imply, however, that Platonism prevailed in the Polish School. I even think that the opposite was the case: most of its members were realists in the ontological meaning of the term. The leading thinker in that respect was surely Leśniewski.

I knew him less intimately than Łukasiewicz, but still had the privilege of being received and was able to have several discussions with him. He was a biggish, strong man, who smoked a large pipe and drank coffee from a very big pot. He was a highly intelligent fellow, perhaps even a genius, but I suspect he knew it and did not feel obliged to work a lot. Anyway he was always ready to entertain me with long speeches about philosophical matters. One of the central tenets in those speeches was always that many people are able to say their head is made out of glass. He named two sorts of such people: those who believe that there is a

God, and those who are addicted to far worse superstition – the belief that they are in classes. I – he used to say – often saw heaps of stones, but never any class of them.

As far as I understand him, his philosophy contained two denials: he refused to admit the existence of any ideal being and of anything except bodies. The latter implied the refusal to admit any category except the "thing", the Aristotelian substance, and the denial that there are any mental entities. This was made more explicit and propagandized by Kotarbiński, but the basic ideas are Leśniewski's. I think that much of his highly unorthodox logic was constructed to serve that sort of ontological refusal.

I might be allowed to mention my own reaction to that philosophy. While thinking that there is no possible explanation of the real without recurring to the ideal (Whitehead), and that real properties are present in the world I admired Leśniewski very much. For one thing he opened the door – so to speak – to something which was missing up to then, namely a logic of parts and wholes – such as was later on sketched by Nelson Goodman and now built up by Peter Simons. It looked to me like a major step forward in logic.

On the other hand, while not following him in his rejection of qualities and relations, I could not refrain from admiring the courage with which he opposed the whole tradition of modern European thought. This was in fact tending to deny the existence of things, not of properties. And I would like to add that the discussions with him had a deep influence on my personal thinking, to such an extent indeed, that most of my basic ontological views were constituted in contrast to his philosophy.

I have little to say about the third great man of the School, Alfred Tarski, in spite of the fact that I met him frequently in California and that he was always exceedingly kind to me. One of his qualities was his patriotism. I do not think anybody could tell all he did for his countrymen after the Second World War. (His wife, Maria, held a high military decoration won during the 1919-20 Russian War).

He was a smallish, rather nervous man, very kind to people, but outspoken in his criticism. I shall not forget how he attacked my dear friend Evert Beth because of a rather indifferent performance of one of the latter's students.

Philosophically Tarski stood nearer to Leśniewski than to Łukasiewicz, being an extreme nominalist – something rather exceptional among the leading logicians of our time. But I think that he

would hardly admit Leśniewski's ontology. Several times I heard him saying that philosophy *is* logic (including, of course, his metalogic) and nothing else. Curiously enough, Łukasiewicz himself sometimes held the same opinion.

Tarski was involved in quarrels with two other logicians. Łukasiewicz and Rudolf Carnap, in both cases about priority of discoveries. As far as the first is concerned, it was not caused by Tarski himself, but by blunders of some others who ascribed to him discoveries obviously belonging to Łukasiewicz. I could not do anything about it, but did my best to smooth the second quarrel. I think I was successful in as far as the two, Tarski and Carnap, did sometimes shake hands.

By pure chance (he was taking part in a congress in the United States when Germans invaded Poland) he escaped death in a gas chamber, as he was of Jewish origin. Initially, he had hard times in the States where people did not seem to realize that they were dealing with a man of genius. But later on he became a sort of king in his field in California. He assisted at all possible congresses and symposia, spending most of the time in pushing his students forward. The result was that many chairs of logic in the state – and not only there – were occupied by them.

During my stay at the UCLA, Austin, a leading member of the Oxonian school, who was visiting Berkeley, paid us a visit. I witnessed a discussion between the liberal, flexible Englishman and the cohort of hard Tarskians composed of our professors. I must say, that Austin defended himself brilliantly – but still, as far as the numbers were concerned, Tarski's domination was obvious.

I come now to another logician whom I knew intimately, Sobociński. An assistant of Leśniewski, he was said to be the only man in the world who really knew everything about his master's logic. A specialist in some very special departments of logic – the search for the shortest axioms and the like – he left a considerable number of results which would surely merit republication.

Let me tell something about him which happened in South Bend in 1956. Sitting at the fireside in his villa, he was bitterly complaining of "all the nonsense which is written today in logical books". "What books?", I asked. "Well, your own books for instance". You can imagine that I asked him to explain. "Of course", he said, "you do assert, like most of the crowd, formulae with free variables, which *is* nonsense". This gives an idea of how true he was to the tradition of rigour of his masters in Warsaw.

Sobociński was also a completely unpractical and most "scholarly" person. During the interview which preceded his nomination in Notre Dame, we had quite a lot of trouble understanding what he meant by saying "Son loves Mary". We asked in vain "whose son?". It appeared finally that not a son, but John was meant. When appointed he asked that the expenses for the transportation of his wife *and of his cat* be paid. And so on. Yet, once in the chair, he proved to be an excellent, brilliant teacher. It is true that he wrote on the blackboard practically all that he said.

The School was by no means homogeneous – its members disagreed sometimes about quite important matters. E.g. for the no-nonsense philosopher Leśniewski the trivalent logic for which Łukasiewicz is famous, was not logic at all, but just a formalistic game.

Yet, there was something common to those logicians: their rigour-clarity of speech and exactness of proofs, both programmatic and practised by all of them. I tried to show in a few instances how great it was. Now the very fact of that prevailing rigour gives rise to two questions. First, where does it come from, what are its historical sources? Second, why did it develop precisely in Poland – so much so, that during that period the Poles were by far more rigorous than most other logicians?

The answer to the first question is surely known: the leading Polish logicians were all pupils of Twardowski, who was himself a pupil of Brentano. Brentano is therefore, not less than G. E. Moore, a forerunner of recent rigorous thought.

What is by far less known, is that Brentano himself was an heir of the scholastic tradition. That tradition, ridiculed by the fine talkers of the Renaissance, among others things precisely because its "subtlety", meaning logic, was coming to its rights among Brentano and his pupils. Analytical philosophy, including its logic was the true neo-scholasticism of those days. The reevaluation of scholastic logic by Łukasiewicz is not an accident.

But why Poland? There were surely several reasons for that. There was the fact that Twardowski, a gifted teacher, was able to educate a great number of philosophers. But it seems to me that the philosophical past of Poland is the background out of which the rise of Polish logic can be best understood. That past has been extremely romantic and irrationalistic. The rise of the Lwow-Warsaw School appears as a reaction against it.

I witnessed one of the last clashes between that modern logical ratio-

nalistic reaction and the old romantic way of thinking and talking. At a philosophical congress Braun, the last representative of a very confused sort of Hegelianism, was rebuked by one of the leading analytical philosophers of the country, Ajdukiewicz. I have never heard anything as harsh as the way poor Braun was attacked. He was told that he was not only an ignoramus, but even a badly educated person who had no right to take part in discussions with civilized people.

Well, the old logicians of the School are all gone. Yet, it seems to me, that their work is not over. Not only because several among their technical achievements have surely an enduring value. More importantly perhaps, their basic attitude will survive: I mean what Łukasiewicz once called "the morals of thought and speech". During my recent visit to Poland, I could observe with pleasure how much is still alive in that country today.

JÓZEF M. BOCHEŃSKI

WHAT HAS LOGIC GIVEN TO PHILOSOPHY?

Both terms appearing in the title of the following remarks are highly ambiguous. As far as 'logic' is concerned, one speaks not only about formal logic, but also about transcendental, material logics, about the logic of emotions – and even about a 'logic of Revelation'. Equally, perchance with philosophy: beside philosophies of the kind that e.g. Ingarden or Kotarbiński were involved with, what is sometimes called 'philosophy' is more or less the 'higher' literary statements of journalists, ideologists, preachers and the like. Given this situation it is hardly possible to discuss seriously the contribution of logic to philosophy, unless one first specifies somehow the meaning of both terms.

The task is not particularly difficult when one speaks about 'logic'. Suffice it to say that what is meant by this is mathematical logic in a broad sense of the word, that is, including not only formal logic but also semantics and general methodology based on it. Two limitations must be imposed here. On the one hand what we mean by 'logic' is not a set of all formalized systems, but a set of such systems that have a logical interpretation. Incidentally, not everything that appears in the *Journal of Symbolic Logic* is logic in the sense of the word introduced above. Without going as far as Quine (for whom even the so-called logic of sets is no logic), it should be pointed out that by 'logic' we shall mean esentially any set of theorems (or rules), similar to those that can be found in the first forty chapters of the *Principia*.

But specifying the meaning of 'philosophy' is not so easy. It should be evident that any meaning, according to which all irrationalisms would be philosophies, is out of the question here: for logic has obviously made no contribution to irrationalisms whatsoever. Admittedly, some irrationalists claim to have another, 'true' logic at their disposal (as for formal logic, Heidegger writes its name in inverted commas), allegedly presented in the *Koran* or some other 'classic text'. However, as was mentioned, those 'true' logics are of no interest to us here. But even if irrationalisms are excluded, there remains quite a large group of various systems, e.g. transcendental ones along with analytic and

Jan Woleński (ed.), Philosophical Logic in Poland, 9–17.

phenomenological ones.

In this situation one might say that academic philosophy is the main issue here. But then the question arises: what kind of philosophy is academic? As is well known, there has been no agreement among philosophers concerning that question.

Consequently, the best solution could be to assume a personal stand here, which can be described as follows: the philosophy that is going to be discussed is a philosophy of the kind pursued by the present writer. What kind of philosophy it is will emerge in the course of the following remarks. Admittedly, this approach is one-sided and free from any pretentions to generality; nonetheless in spite of this it may make a good point and serve as at least one person's testimony that relates the role that logic played in his attempts to philosophize. Moreover, this writer suspects that his case is not an isolated one. Many other philosophers could say the same thing about themselves.

Now, in my opinion, logic, conceived in this manner, has contributed to the advance of philosophy in three ways. First, as a *paidagogos*, an educator. What I understand by this is that logic gave the philosopher who had had some experience of it, an example, a pattern of some kind of thinking and speaking – if we can put it like this – that it educated him in this respect. Logic has also made a major contribution to the development of philosophy as its *organon*, a tool, as the ancient Aristotelians considered it to be, namely by providing philosophers with tools of thinking which were more perfect than those that prevail among people who do not know logic. Finally, logic has given much to philosophy as a *meros*, a part of philosophy, as the old Stoics understood it. What I mean by this is that logicians' studies, carried on in their own field, validated some claims and made certain notions (which evidently belong to the realm of philosophy) more precise.

To repeat, one can thus speak about a threefold contribution of logic to philosophy: as a philosopher's *paidagogos* and *organon* and as a *meros* of philosophy itself.

I.

Łukasiewicz once said that logic was the ethics of speech and thought. This claim is certainly true in the sense that logic provides the ideal of rationality. For it is said that someone speaks and thinks rationally, that is reasonably, when first, he uses only expressions whose meaning

he can explain, and second, he puts forward only those propositions he can justify. An axiomatic system properly constructed is the model of rationality of speech and thought. In such a system there is no room for unclear expressions, arbitrarily accepted statements and enthymematic proofs. It is precisely logic that teaches the theory of the axiomatic system; and also within logic can be found the most correct application of this theory.

So much for theory. In practice, the difference between philosophers who learnt mathematical logic and others is often truly striking. Anyone can easily come to the conviction that this is the case when he compares texts of allegedly 'great' philosophers like Hegel or Kant with works of thinkers who know modern logic. Therefore I shall not quote examples; instead I should like to report an impression that I repeatedly get at international congresses of philosophy. In recent times circumstances have allowed several, sometimes even more than a dozen philosophers from Poland to take part in them. Now, what struk me was how much these Polish philosophers differ from others in what – along with Łukasiewicz – I would call honesty and decency in speech and thought. They speak orderly, the things they say have, as the Germans would put it, 'hands and feet'. Why is that so? I think one of the reasons, if not the main one, is the fact that Polish philosophers have been educated in an environment saturated with modern logic. It is a great educator of thinkers. And this is, if I am not mistaken, its first contribution in this respect.

II.

As far as the role of logic as an *organon*, a set of a philosopher's tools is concerned, there is a common belief that formal logic is only a set of conceptual tools, needed for correct reasoning. For various reasons this claim cannot be maintained any more. First, because one who uses simple arguments, and this is often the case of the philosopher himself, has absolutely no need of theoretical logic, even Aristotle's syllogistic, in order to reason correctly and check the correctness of his arguments. Incidentally, if logic can be useful for the analysis of more complicated arguments, its major role in this area consists in providing an excellent tool of the analysis of concepts rather than arguments.

So we are right to say that modern logic conceived as a set of conceptual tools may give much to philosophy – and I think it has – in relation to the analysis of complex arguments, with the analysis of concepts in

particular.

Starting with arguments we find that complex arguments appear relatively rarely in philosophy; they occur above all in metaphysics. By 'metaphysics' (as opposed to ontology) I understand a discipline dealing with non-phenomenal objects (in the Husserlian sense), which are somehow linked with the whole of human experience, that is 'objects' of Kant's 'ideas': soul, world and God. One gets the impression that it is only or almost only here that truly complex philosophical arguments can be found.

Now logic as a tool has given much to philosophy in this respect. Admittedly, we do not have many metaphysical studies in which modern logic has been used, but a few can be quoted here. One of them (and possibly the most important) is Jan Salamucha's analysis of the proof *ex motu* for the existence of God. It is, in my opinion, one of the most significant metaphysical studies that appeared in print in the 20th century. Anyone who compares them with classical reports of the same proof, e.g. that of Garrigou-Lagrange, will easily see that the advance achieved with the help of logico-mathematical tools by Salamucha is immense. For it is in his work that we first find a full analysis of this proof; he is also the first to have shown that the Thomist proof *ex motu* in this form is correct; furthermore, Salamucha is the first metaphysician who was able to enumerate all the premisses of the proof; finally, he is also the first to have drawn attention to the possibility of various interpretations. This is hardly surprising since Salamucha had at his disposal, amongst other things, two logical tools unknown to his predecessors: the technique of formalistic axiomatization and sequence logic (indispensable in every proof for the existence of God, but never mentioned by Garrigou-Lagrange or other philosophers of his time).

Admittedly, one could say this is not much. But Salamucha's example clearly shows that applying logic contributes to the progress of the analysis of complex philosophical arguments. The same example allows us to understand something that the majority of old philosophers seem not to have understood, namely that metaphysical arguments are exceedingly complex and difficult. Consequently, the application of logic as a tool of these arguments resulted in common mistrust in metaphysics among those philosophers who knew modern logic – not simply because it, as Kant would hold, transcended the limits of experience (for every theoretical statement transcends them after all), but because this domain is so extremely complex and difficult.

At the same time modern logic as a set of tools for the analysis of reasoning opens new horizons before metaphysics – the possibility of pursuing it in a scientific manner, that is orderly and careful. It opens them because e.g. Kantian proofs against the possibility of metaphysics are, in the light of logic, not only incorrect, but often confused.

So much for philosophical arguments. But logic as a tool has given much to philosophy first of all in the domain of the analysis of concepts. Here are four examples:

(1) Russell's analysis of the concept of existence (the theory of descriptions): One may agree or disagree with its author, but one can hardly deny that the discussion it stimulated would not have been possible, if Russell had not employed logico-mathematical tools.

(2) The dispute over universals with Quine's ontological criterion: Once again, one may accept or reject the American logician's standpoint, but one cannot possibly deny that the discussion on universals has been thoroughly renewed due to the application of logico-mathematical concepts and tools (the concept of the variable and the quantifier).

(3) For the first time in history the concept of truth has been correctly defined by Tarski with the use of a highly advanced logico-mathematical apparatus. Because this definition is also of vital importance when we talk about the contribution of logic as a part of philosophy, we shall come back to it later.

(4) Finally, the concept of analogy, central not only to scholastics, but possibly to the majority of contemporary sciences as well (including logic itself), has been scientifically analysed for the first time, owing to Drewnowski's suggestions that analogy was merely an isomorphism.

The list of concepts analysed in this manner could be extended. The present writer could nearly always employ logical tools in his modest philosophical investigations, and, through using them, achieve interesting results. By way of example, let me quote a fragment of the analysis of a free society of the year 1985.[1] We call a society 'free' when its members are free; so the concept of the free society presupposes the

concept of the free individual, i.e. the concept of individual freedom. Now the latter is, from the point of view of logic, a binary relation: x is free in the domain y. For example, I am (politically) free to choose the direction I am taking, to Bern or Milan, but I am not free to choose the side of the street on which I shall drive because I am not allowed to drive on the left side. Thus the main structure of individual freedoms is represented by the formula '$W(x, y)$', where 'W' is a sentence-building functor, whose first argument is an individual name, and the other a class name. The concept of the free society will thus be a generalization of this formula. How many such different generalizations are there? Mathematical logic teaches that there are six of them, which together with their negations makes twelve. Starting with '$(x, y)W(x, y)$': 'every x is free in all domains' (which seems to be the definition of the anarchic society), we progress through to '$(x, y)\neg W(x, y)$': 'no 'x' is free in any domain' (which probably corresponds to extreme totalitarianism, which is impossible anyway). There are ten simple indirect structures between these extremes, e.g. '$(\exists y)(x)W(x, y)$': 'there exists at least one domain, in which every x is free'. Most of the actually existing societies have complex structures which may be represented by the product of various simple functions. Thus the application of two fundamental concepts of mathematical logic, namely, the concept of the dyadic functor and, connected with that, the concept of double quantification, has made this analysis posible. Not only a man in the street, however, but old logicians, too, had no idea about these concepts whatsoever.

Thus formal logic has given much to philosophy as a tool of thinking.

III.

Finally, logic is also (contrary to the one-sided beliefs of old Aristotelians) a *meros*, a part of philosophy. Formal logic is nothing but an ontology of some kind, and if some discipline belongs to philosophy, it is surely ontology. One of the reasons for adopting this view is the fact that logic unexpectedly gave relatively much, not only as an educator and as a tool of philosophy, but by the fact that some of the results of purely logical investigations have proved to be a contribution to philosophy, sometimes a very important one.

Here are some examples:

(1) Finding the antinomy of the class of all classes forced Russell to put forward a claim that there is no universal class having as its elements

both things and properties, relations, etc., i.e. that reality is many-layered, as it is proper to say in the ontological language. This belief has been expressed in the form of the so-called theory of types. Now, this theory is nothing but a revival of a medieval view, according to which the expression 'being' is not univocal, and no expresion could be used for denoting simultaneously both things and properties in an univocal way. Moreover, the introduction of the theory of types has brought about considerable difficulties in logic and mathematics because it demanded the acceptance of the fact that there existed e.g. not only one three, but an infinite set of different threes, one for each type. To find a way out, Whitehad and Russell introduced *systematic ambiguity*, allowing the use of a single expression to denote ambiguously in different types. Through this they discovered anew (probably unaware of it) the scholastic theory of analogy, together with its medieval name. For in the Middle Ages analogy was also called the '*aequicicatio a consilio*', i.e. 'systematic ambiguity'.

One could think that since this is the case, we do not have to do with true progress. This would be a mistake, though. Contrary to medieval philosophers, we have today, thanks to mathematical logic, a proof that 'being' cannot be conceived without contradiction as an univocal name. Thus logic has permitted the final settlement of this philosophical dispute, carried on for centuries.

(2) The second example is the above-mentioned definition of truth of Tarski. Pilate's question 'What is truth?' is certainly a philosophical issue. Now the Polish logician has succeeded in defining correctly the Aristotelian concept of truth with the use of complex logical and semantical tools. Tarski has not proved, of course, that the term 'truth' could be used only in this sense because it cannot be proved. But by providing a correct definition of the concept of truth, free from contradictions, he has made a major contribution to the advance in this area. At present the Aristotelians are much better off than the adherents of other concepts of truth e.g. the transcendentalists and the pragmatists. For up till now, none of them has been able to define his concept of truth in a manner that would even slightly resemble the exactness and clarity of Tarski's definition.

(3) Gödel has proven (and the word 'proven' is worthy of emphasis here) that for each sufficiently strong axiomatic system there exists at least one true but undecidable statement, i.e. a statement that neither can be proved in the system nor does it stand in contradiction with it. So

he has proved that there could not be a 'general' all-embracing system. Few philosophers have noticed I dare say that this theorem led to results of great philosophical importance. For, if Gödel's theorem holds true for his relatively weak system, how much more it does for stronger systems. What follows from it is that all-embracing systems such as Hegel's philosophy are logically impossible. And again we have an important philosophical conclusion, made possible by logicians' work in their own discipline.

(4) As is well known, modern logicians create axiomatic systems. Now an axiomatic system properly constructed includes, amongst other things, the enumeration of primitive terms used in it, i.e. the most abstract concepts, in other words categories, according to the ontological interpretation. Even if such a system has not been constructed quite rigorously, as in the case of the *Principia*, yet this allows us to enumerate easily those primitive terms or concepts. Now a group of these words is in normal systems of mathematical logic strikingly similar to the correctly understood list of fundamental concepts of Aristotle's ontology.

Here are for example the primitive expressions of the *Principia*:

1. individual, subject; symbols 'x, 'y', etc.;
2. property; symbols 'φ', 'ψ', etc.;
3. relation; symbols 'Q', 'R', etc.;
4. existence; symbol '$(\exists)$';
5. fact ('p', 'q', etc.);
6. one constant relation (Sheffer's functor)
7. relation between a property (relation) and a subject (subjects), as for example in 'φx' – a concept for which there is no symbol.

All these concepts save fact (5) and the equivalent of Sheffer's functor (6), play a vital part in Aristotle's ontology. 1, 2 and 3 are his categories (*ousia, poiotes, pros ti*). Existence has not been mentioned in the list of the categories, but it is after all one of the basic concepts of Aristotle's ontology. The most interesting case is probably concept 7, which is strikingly similar to another central concept of the Stagirite, namely to the concept of the relation between act (*energeia*) and potency (*dynamis*), or form (*morfe*) and matter (*hyle*). The similarity to the latter relation is even expressed in the fact that the *Principia* treat the symbol of property ('φ') as a constant, that is a sign of something definite, and

the symbol of subject ('x') as a variable, that is a sign of something indefinite, as in the case of form and matter.

Logicians' work has thus given two things here. On the one hand it has allowed us to simplify and complete Aristotle's list of basic concepts of ontology. On the other hand it suggests that scientific inquiry in the field of logic, i.e. ontology is possible only if the intuitions of the founder of both disciplines are principally followed; that e.g. other lists of categories may be interesting, but they are void of any practical value. Their adherents were not able to construct anything that would equal the systems of modern logic in their exactness and richness of content.

So the situation is similar to that in the domain of the above-mentioned concept of truth: Logic has made an important contribution to the advance of philosophy.

NOTE

[1] See J. M. Bocheński: 1986, 'The Concept of the Free Society' *The Monist* **69**(2), 207-215.

ANDRZEJ GRZEGORCZYK

CLASSICAL, RELATIVISTIC AND CONSTRUCTIVIST WAYS OF ASSERTING THEOREMS

1. ASSERTION AS THE FUNDAMENTAL COGNITIVE ATTITUDE TOWARDS PROPOSITIONS

Most people who use formal systems, and in particular logical systems, feel that because these systems lack any connection with colloquial speech, they also lack the function of expressing beliefs, performed by ordinary language sentences uttered with a serious intent. Those systems become merely a technical tool for further technical scientific constructions. Thus, the whole of deductive science becomes largely a technique, i.e. a kind of creative activity, which consists not in discovering an objective reality, but rather in constructing one. It affords a rather subjective pleasure of constructing complicated entities out of simple elements. The only way to evaluate the construction objectively is to employ the criterion of its applicability to further constructions, or perhaps the criterion of the author's ingenuity in devising an original and imaginative contrivance.

Some authors who have a strong intuitive feeling for the subject-matter of their discipline inveigh sometimes against the technical character of deductive sciences. Such was perhaps the starting point of both intuitionistic and constructivistic criticism of classical logic. The initiators of these schools began by asking searching questions about the contents of formal systems and by inquiring into the grounds for asserting the propositions contained in formal systems.

The basic intellectual attitude one can take towards a formula one understands and regards as meaningful is either to accept it as the expression of one's beliefs and assert it, or else to reject it. An asserted formula (one which is asserted to be true) is a formula which we directly link with our cognitive experience and which we accept as the expression of our beliefs about reality. When we speak about assertion, we are voicing a call to take language seriously, to treat verbal constructions as expressions of beliefs rather than as expressions of free creation.

Jan Woleński (ed.), Philosophical Logic in Poland, 19–31.

I wish to follow this direction and to discuss both types of assertion and the actual ways in which theorems are asserted in mathematical science.

2. METHODS OF ASSERTION

In common parlance, logical calculi can be given either axiomatically or semantically (i.e. by matrices). This basically correct distinction stands in need of emendation. Let us start with matters of terminology.

In most cases, theories, calculi and systems are potentially infinite sets of propositions. No one is therefore capable of holding them in his mind in a single intellectual vision. Usually, one who believes the propositions of a system is in the possession of some method of assertion.

Broadly speaking, one can distinguish authentic methods of assertion and those based on authority. The former are used by the authors of the system, and those who have adopted the authors' method and use it themselves. The latter consist in asserting what someone else has asserted, whom we then call an expert in a given discipline.

In what follows we shall be concerned only with authentic methods of assertion, attempting to single out some of their formal characteristics. To facilitate this task, we shall assume that propositions (asserted and unasserted) are formulas, i.e. inscriptions with a well-defined structure. We will be interested in the potential arguments that could justify the methods under discussion.

One can distinguish three fundamental methods of assertion: (1) verification, (2) inference (deduction), and (3) semantic construction of asserted propositions.

2.1. *Verification*

This is the simplest method, but its scope is rather limited. It consists in performing simple empirical operations, which enable us to find out in a finite number of steps whether a given formula is a theorem or not. Simple empirical formulas that ascribe a directly verifiable property to some object are verifiable in this sense. For example the formula: "My table is rectangular give or take 1 cm" can be verified by making a simple measurement. In the arithmetic of natural numbers, atomic propositions such as $238 + 764 = 982$ or $43 \times 27 = 1161$, are likewise

verifiable. They can be verified by carrying out a computation with the help of a suitable algorithm and by comparing the results. Formulas of classical propositional calculus are also verifiable. However, it seems that as far as mathematical and logical theories are concerned, the verifiability of their theorems is a secondary property. These theories are not adopted on the basis of a method of verification, but because of the intuitive appeal of their axioms, or because of their agreement with some accepted semantics. It then turns out that certain mathematical and logical theories are decidable, i.e. that their is a mechanical method of deciding in a finite number of steps which formulas are theorems of the theory and which are not. The case of atomic propositions of natural number arithmetic is somewhat different. One may claim here that the basic operations on natural numbers, that is, addition, multiplication, subtraction, raising to a power, etc., and also the basic relations between numbers, such as identity, difference and order, are given to us by algorithms enabling us to perform these operations on numbers expressed in some representation. Thus, the assertion of atomic propositions need not rely on any wider set of propositions, because it can be based on such an algorithmic method of verification, independently of the rest of arithmetic. In a way, the atomic propositions of arithmetic form a kind of base, consisting of empirical theorems about natural numbers. They furnish us with the fundamental facts of number theory, which is merely their generalization. Definitions of operations adopted in the theory are meant to capture theoretically our experience of working with computation algorithms.

2.2. *Inference*

Deducing propositions from other propositions is obviously one of the fundamental methods of passing from one asserted proposition or rule to another. We assert a proposition as soon as we manage to infer it from those that have already been asserted.

A theory (deductive system) is defined as the set of all propositions that can be derived from axioms.

From the point of view of constructiveness, the method of assertion by inference is characterized by the fact that axioms are usually those theorems whose assertibility is indisputable. Axioms are taken to be intuitively obvious. Rules are similarly regarded as directly evident. To argue with someone about axioms and rules we would have to appeal

to some other method of assertion than inference. Obviously, if axioms are to be asserted on the basis of intuition, the set of axioms must be directly surveyable. This is possible only if it is finite or is reducible to a finite number of schemes. Therefore, the method of assertion by inference must be based on a decidable (recursive) set of axioms and rules. If axioms and rules are treated as self-evident, it must be possible to recognize them directly. We must have a method of deciding which formulas are axioms and which are not, and a method of checking which relationships between formulas follow the rules and which do not. Formal decidability of the property of being an axiom is an immediate consequence of its intuitive assertibility, but of course it is not sufficient for assertion. A set of propositions unequivocally rejected may equally well be recursive, i.e. its elements may be recognizable on the basis of simple algorithmic operations in a finite number of steps. If a theory (i.e. a set of asserted propositions) is by definition based on a recursive set of axioms and closed with respect to recursive rules, then the whole theory is recursively enumerable; yet, it does not have to be recursive (decidable). Therefore, we may have no method of deciding which formulas belong to the theory and which do not; we do, nevertheless, have an effective method of generating asserted propositions belonging to the theory. The successive writing out of longer and longer proofs is a method of automatic generation of new propositions, which eventually is guaranteed to generate any given proposition that belongs to the system. This, however, does not in general provide us with a method of checking which formulas are theorems of the theory and which are not. For, if up to a certain point we have not yet generated either a formula A or its negation, then we cannot know if A would be generated, were we to continue the process. However, if the theory is complete, i.e. if for any formula A, either A or its negation belongs to the theory, effective generation gives us a decision method, because, assuming that we continue to generate theorems for a sufficiently long time, we will eventually obtain either A or the negation of A. In other words, we have a method of determining whether A is a theorem or not.

2.3. *Semantical construction of asserted propositions*

We have seen that the deductive method consists in arriving at new asserted propositions by deducing them from other, already asserted ones. The semantical construction also relies on previously asserted

propositions, but it does so in a completely different way. Characteristically, it proceeds from the assertion of short component sentences to the assertion of long and complex ones. In the semantical construction we consider the question: What does it really mean to assert a complex sentence (a conjunction, alternation, implication, negation, universal and existential quantification"? Different semantical systems provide different answers to this question. I shall investigate their types.

A classification based on the criterion of whether assertion admits of degrees immediately suggests itself. It also seems natural to distinguish absolute assertion and assertion relativized to the conditions of asserting. Assertion is always relativized to the atomic base, for which the mode of assertion is assumed to be given. The specification of the conditions of asserting complex propositions is important from the point of view of a logical system. In building a logical system we aim at finding such general schemes of complex sentences that any proposition falling under one of these schemes is an asserted proposition, independently of the way in which atomic formulas are asserted and the circumstances under which the assertion is made. We will now survey a few semantical constructions of an asserted sentence and attempt a comparison.

3. A COMPARISON OF THREE SEMANTICAL THEORIES OF ASSERTION

3.1. *Classical assertion*

In terms of the above-mentioned classification, classical assertion is absolute, i.e. not relative to the conditions of assertion, and does not admit of degrees. According to the classical view, to be asserted means the same for a proposition as to be true. Thus, there are only two kinds of propositions: asserted (to be true) and rejected (as false). Moreover, if a sentence is asserted to be true, it is asserted to be such independently of the circumstances under which the assertion is made. A sentence asserted to be true today is regarded as one which should be asserted by everybody and at any time, regardless of whether the proposition pertains to events past, present or future, verifiable or not. The classical notion of truth is linked to a kind of philosophical stance which was perhaps typical of classical ontological philosophy, both the Aristotelian and Platonic schools, which agreed in affirming the existence of eternal laws. The law of the excluded middle, valid in classical logic, was a

fundamental principle of both these ontologies. The classical conception of an asserted proposition rules out any relativization to a method of assertion.

In comparing the different definitions of an asserted proposition we will be investigating primarily the inductive steps of these definitions. In the classical case they take the following form:[1]

(1) we assert 'A & B' iff we assert 'A' and we assert 'B'
(2) we assert 'A or B' iff we assert 'A' or we assert 'B'
(3) we assert 'If A then B' iff we either do not assert 'A' or we assert 'B'
(4) we assert 'Not A' iff we do not assert 'A'
(5) we assert 'For all x, $A(x)$' iff for any elements a in M we assert '$A(n(a))$'
(6) we assert 'For some x, $A(x)$' iff for some elements a in M we assert '$A(n(a))$'

where $n(a)$ is a previously chosen name of the element a or the so-called valuation.

It may seem strange to someone not acquainted with semantical constructions that the assertion of propositions containing connectives is defined by means of the very same connectives used in the metasystem. The same, however, holds true of the deductive construction. We use implication in the metasystem in order to express the fact that the system is closed with respect to, say, the rule of modus ponens.

If the definition of assertion is relativized to a valuation, or, in other words, if it is a definition of satisfaction, then we assert a proposition if we assert it under every valuation. However, if the system is used to describe a domain whose every element has a name, then relativization to valuation is not needed. The inductive definition can be a definition of an asserted proposition from the start. The arithmetic of natural numbers provides an example:

(7) Atomic propositions of the form '$n+m = k$' and '$n \times m = k$', where n, m, k are names of particular natural numbers, are asserted iff they are algorithmically verifiable

Then the inductive conditions (1)–(6) given above, taken together with (7), constitute a full definition of an asserted proposition without any relativization to a valuation. $n(a)$ is then taken to be a numeral naming the number a in a given notation. M is of course the set of all natural numbers.

Perhaps it is worth stressing that the above-defined notion of an arithmetical proposition asserted to be true can intuitively be purely syntactical. We do not need to speak of the domain of natural numbers. It is enough to have the notion of an arbitrary numeral. We can then express conditions (5) and (6) in the following form:

(5') we assert 'For all x, $A(x)$' iff for any numeral n we assert '$A(n)$'

(6') we assert 'For some x, $A(x)$' iff for some numeral n we assert '$A(n)$'

Algorithmic verification of an atomic proposition can likewise be regarded as a purely syntactical procedure.

The definition given above could be criticized precisely for making the notion of an asserted proposition identical to the notion of truth understood as conformity to reality, whilst assertion should be something less than truth, in the sense that we can assert only what can be effectively verified to conform to reality. The notion of assertion is different from the notion of truth and it should have a different definition. Even assuming that we use classical logic in the metasystem we should not be obliged, for every proposition A, to assert A or to assert not-A. This argument leads to relativistic conceptions of assertion.

3.2. *Relativistic conceptions of assertion*

It seems, therefore, that these conceptions are based on the view that assertion of a statement is a psychological act, and depends on various conditions. This dependence need not be seen as a deficiency (a kind of inevitable opportunism), but rather as a necessary caution justified by methodological reasons. To describe the dependence of assertion on conditions in a most general way, we will treat the set of conditions influencing assertion as constituted in a certain way by mutual relations. We may take them to have linear, or perhaps partial order structure, or, even more generally, to be a topological space. It also appears that the space of conditions is dense. We can almost always think of intermediate conditions. The conditions of the assertion of atomic propositions seem to constitute an open set. If we assert certain propositions under given conditions, then it is usually easy to imagine similar conditions, more distant in a certain respect, under which we would also be willing to assert the same proposition. At the same time, we can imagine a border,

beyond which we would no longer be entitled to the assertion of the proposition. The indications of measuring equipment are an example of assertion conditions; such indications are never precise, and it is always possible for the indicator to give a slightly different reading without our revoking the result of the measurement.

To continue this train of thought, we might stipulate the following inductive conditions to define the relativistic notion of assertion:

(11) we assert 'A & B' under the conditions w iff we assert 'A' under the conditions w and we assert 'B' under the conditions w

(12) we assert 'A or B' under the conditions w iff we assert 'A' under the conditions w or we assert 'B' under the conditions w

(13) we assert 'If A then B' under the conditions w iff there is an open neighbourhood G of the conditions w (in the space of conditions) such that, for every element u in G, either we do not assert 'A' under the conditions u, or we assert 'B' under the conditions u

(14) we assert 'Not A' under the conditions w iff there is an open neighbourhood G of the conditions w, such that, for every element u in G, we do not assert 'A' under the conditions u

(15) we assert 'For all x, $A(x)$' under the conditions w iff there is an open neighbourhood G of the conditions w such that, for every element u in G and every element a in M we assert $A(n(a))$ under the conditions u

(16) we assert 'For some x, $A(x)$' under the conditions w iff for some element a in M we assert $A(n(a))$ under the conditions w

In the case of arithmetic, the conditions (15) and (16) can be replaced with (15') and (16') in a way analogous to the classical case. (Cf. (5), (6) and (5'), (6')).

Under the relativistic notion of assertion we make one further step, namely we define the absolute proposition:

(17) we assert A absolute iff we assert A under all conditions

The theorems of arithmetic or logic are precisely the propositions that are asserted absolutely (or, in the case of logic, propositions asserted absolutely for every model M).

It is easily seen that for many spaces of conditions the set of propo-

sitions asserted absolutely is identical with intuitionistic logic. If S is a space of conditions and assertion is defined as above, then we can define a valuation of expressions in the space S:

(19) G is the value of the expression A in the space S iff G = the set of all conditions under which we assert the expression A

Abbreviating this to

(20) $G = \text{Val}_s A$

we can deduce the following identities from the conditions (11)–(16) imposed on assertion:

(21) $\text{Val}_s \ulcorner A \vee B \urcorner = \text{Val}_s \ulcorner A \urcorner \cup \text{Val}_s \ulcorner B \urcorner$

(22) $\text{Val}_s \ulcorner A \wedge B \urcorner = \text{Val}_s \ulcorner A \urcorner \cap \text{Val}_s \ulcorner B \urcorner$

(23) $\text{Val}_s \ulcorner A \rightarrow B \urcorner = \text{Int}(S \backslash \text{Val}_s \ulcorner A \urcorner \cup \text{Val}_s \ulcorner B \urcorner)$

(24) $\text{Val}_s \ulcorner \neg A \urcorner = \text{Int}(S \backslash \text{Val}_s \ulcorner A \urcorner)$

(25) $\text{Val}_s \ulcorner \forall x A(x) \urcorner = \text{Int}(\cap_{a \in m} \text{Val}_s \ulcorner A(n(a)) \urcorner)$

(26) $\text{Val}_s \ulcorner \exists x A(x) \urcorner = \cup_{a \in m} \text{Val}_s \ulcorner A(n(a)) \urcorner$

These identities define valuation in intuitionistic logic.[2] Conversely, if the intuitionistic valuation of expressions by open sets in some space S satisfies conditions (21)–(26), then (providing that the space S consists of points) we can define assertion at a point in the space S:

(27) we assert A at point w iff w is an element of $\text{Val}_s A$

Assertion defined by (27) satisfies conditions (11)–(16) imposed on the notion of condition-relative assertion. Hence, provided the space of values can be treated as consisting of points (which is usually the case, due to the possibility of representing distributive structures by algebras of sets), the construction of matrices for the so-called many-valued logics can be carried out in such a way that multiplicity of assertion conditions replaces multiplicity of values. There is no need to admit truth values other than truth and falsehood; instead, we can say that the given logical system consists of propositions asserted absolutely, where assertion is relative to conditions, and where the conditions constitute some given space.

It seems to me that a system will be more readily regarded as ad-

missible, or even as valuable, if it is thought of as being concerned with relativization to a space of conditions, rather than if it is taken to imply the existence of degrees of assertion itself (which would then constitute a space). Perhaps a linear ordering of the degrees of assertion can still seem relatively natural (stronger or weaker assertion), but a more complicated structure of assertion (or: the structure of the space of truth values) seems devoid of empirical meaning. On the other hand, a structure (or a topological space) of the conditions of assertion, even quite a complex one, can be given an empirical interpretation. (One possibility is to interpret relativized assertion as assertion relativized to time; cf. A. Grzegorczyk [1]).

When comparing classical and relativistic assertion, I spoke in favour of the latter. Without taking sides, one should make the following distinction. There are two different stances in philosophy, or two independent notions of value: value understood ontologically and value understood empirically. It is worthwhile to present this distinction in greater detail:

Value	Ontological definition	Empirical definition
truth	that which conforms to reality	that which is asserted in every situation
beauty	that which has internal harmony, or embodies an eternal model	that which appeals to us independently of any condition
goodness	that which contributes to the enlargement (enrichment) of someone's life, or embodies natural law	that which is the expression of love, or of the unconditional voice of conscience

Thus, there is an essential difference between the ontological and the empirical way of doing philosophy. Both ways seem justified, but they can lead to different conclusions and to different logical calculi. The ontological way of thinking results in a theory of the structure of objects independent of their cognition. In logic, this leads to classical formal calculus. The empirical way of doing philosophy, on the other hand, leads to intuitionistic, or constructivistic formalism. It takes into

account the manner in which we come to recognize the value, and it makes the very ascription of value dependent upon that manner.

3.3. *Constructivistic assertion*

Constructivistic assertion is relativized to a method of assertion. By method I mean a finite sequence of steps, which would lead to the desired result. In the case of deduction, the steps in question consist in the transformation of expressions in accordance with certain rules; such a sequence of steps is called an algorithm. As remarked above, in the case of arithmetic there are algorithms for checking the validity of atomic propositions. In order to define assertion, we begin by stipulating the following condition:

(30) we assert A (atomic) on the basis of algorithm α iff α enables us to verify A

We then proceed inductively:

(31) we assert 'A & B' on the basis of algorithm α iff α determines two algorithms α_1 and α_2, such that we assert 'A on the basis of α_1 and we assert 'B' on the basis of α_2

(32) we assert 'A or B' on the basis of algorithm α iff α determines two algorithms α_1 and α_2, such that α_1 selects either the expression A or the expression B, and α_2 enables us to assert the expression selected by algorithm α_1 (in other words, algorithm α_2 is the basis for asserting the expression selected by algorithm α_1)

(33) we assert 'If A then B' on the basis of algorithm α iff for every algorithm β, algorithm α transforms β into a new algorithm $\alpha(\beta)$, such that if we assert 'A' on the basis of algorithm β, then we assert 'B' on the basis of algorithm $\alpha(\beta)$

(34) we assert 'Not A' on the basis of algorithm α iff for every β, if we assert 'A' on the basis of algorithm β, then algorithm $\alpha(\beta)$ leads to contradiction

(35) we assert 'For all x, $A(x)$' on the basis of algorithm α iff for every element m in M, $\alpha(n(m))$ is an algorithm, such that we assert $A(n(m))$' on the basis of algorithm $\alpha(n(m))$

(36) we assert 'For some x, $A(x)$' on the basis of algorithm α iff α determines algorithms α_1 and α_2, such that α_1 determines a name $n(m)$ designating an element m in M, and we assert

'$A(n(m))$' on the basis of algorithm α_2

(37) we assert a formula A to be a theorem iff there is an algorithm α, such that we assert the formula A on the basis of algorithm α

The algorithms referred to in this new formalization are taken to be numbers of partial recursive functions. Logic and arithmetic based on constructivistic semantics are similar, though not exactly the same, as intuitionistic logic and arithmetic.

If we compare the semantics relativized to conditions and the semantics relativized to the method of assertion, we will notice an essential difference between the respective definitions of a theorem (a formula asserted to be a theorem). In the former case, to be a theorem means to be asserted under arbitrary conditions. In the latter, it is enough that there should be just one method which would enable us to assert a formula, asserted to be theorem. Thus, in this definition, the space of methods does not play such an important role as the space of conditions in the semantics discussed earlier.

The conditions (30)–(37) represent only one version of assertion on the basis of a method, known under the name of recursive realizability (Cf. S. C. Kleene [2]).

Therefore, in its basic form, constructivistic assertion does not admit degrees, either. The above comparison was intended to show that all known versions of assertion can be treated as relativized to various things, rather than admitting degrees. Instead of operating with many values, we may use relativization; this opens many possibilities for the creation of various semantical systems and also gives greater scope for philosophical interpretation.

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NOTES

[1] Cf. A. Tarski [4].

[2] Cf. H. Rasiowa and R. Sikorski [3], p. 422. "Int" denotes interior in the topological sense.

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H. HIŻ

SEMANTIC NICHES OR A LOGIC FOR HISTORIANS

Il n'y a que la parole, the rest is linguist's construction
(Zellig Harris)

Historians deal partly, even mostly, with what people said. They primarily ask not whether what people said was true, but whether it is true that they said it. They want to know how people were understood by their listeners, by their contemporaries, by the succeeding generations, including our own. And what responses, verbal or other, their utterances have received. They try to place people's utterances in a suitable speaker-listener relation within the setting of place and moment. It is required that historians tell us not just the words used by the speaker but also how these words were received and understood. If so, historical reports which deal with verbal activities, must state that a speaker using a form of speech said to someone that so and so. There is a semantic relation between the speaker, the listener, the form, and what is said. How an utterance was heard, understood and reported is historically more important than how it was actually uttered or intended by the speaker. The speaker seldom reports his own speech and, if so, unreliably. Someone else reports his utterance. A major preoccupation of historians is to establish the reporter's credentials. The utterance may be perceived as a statement, a hypothesis, a question, a command, a reproach, a denial, a lie, or as in some other possible mode of saying. The mode of the utterance may be given by the way the speaker said it. Sometimes, the reporter adds his own impression of the mode. If a person A utters a form α to a listener B and if his utterance is in a mode m, then a reporter C may tell us about it in his own mode n. Thus I suggest as a general schema for communication:

(1) C reported in mode n that A using the form α, to B, in mode m said that p

We distinguish what was said from how it was said. It was said that

Jan Woleński (ed.), Philosophical Logic in Poland, 33–59.

p by the use of the form α. But the form α was uttered together with a gesture, facial expression, strength of voice, facing somebody. In some cases it may be difficult to keep the distinction between the form and those other circumstances that are significant for communication. For instance, if A uses *Today is Tuesday* with a stress on *today*, we may describe the event either by saying that the speaker used the form *STRESS (today) is Tuesday* or by reporting: A reproachfully uttered *Today is Tuesday*. In the first case *STRESS (today)* is a part of the form, not a mode. In the second, reproachfully is a mode and not a part of the form. In case of the report: Laughing, A uttered *Today is Tuesday*, laughing is a mode and cannot be rendered in α itself, if only for lack of a suitable spelling for speech with laughter. Syntactically, a mode is a modifier of *say*, or of another container verb in place of *say*, whereas an intonation pattern may be a component part of the contained sentence.

The listeners decide for themselves what was said. They decide both the form and the content. No matter what sounds the speaker emitted, the listeners heard them as English, or French, or Polish, etc. utterances. This paper deals with the physical and syntactic aspects of the dispatch α by A as organized by B and by C, and with the semantic reception by B of the message that p. Uttering α, A says to B only what B gets from the utterance, no matter what were the "intentions" of A. We find semantic facts in reactions of the audience. If C is present at the scene, he makes a judgement that A addressed B, even if in the very utterance α there is no indication to whom A spoke. Neither will there be much indication in α of who A was. This is supplied by the observer C. His report is a source for historians. C himself is a journalist rather than a historian. C may be identical with B, in which case the report is autobiographical. A is "serving" an utterance for others to understand. Each of the recipients of the message that p "processes' it within his own comprehension. Accumulating and sorting such data, the listener will eventually form an idea of the speaker's opinions. I will call this idea the listener's niche for the speaker. A semantic niche is somebody's hypothesis about what somebody else thinks, knows, considers, understands. It is, therefore, a binary relation between a person holding the hypothesis and a person whose thoughts the hypothesis describes.

When somebody, A, says to somebody else, B, that p, several things are happening. First, B receives a message that p. He may take it seriously and consider that he learned that p. But depending on the mode in which B perceived the utterance by A, he may receive it with

reserve, disbelief. In any case, he understands what was said, and understanding consists in being able to process the message, e.g., to repeat it, to draw consequences, to know how to oppose it, to compare it to other heard utterances, etc. The message which B receives is about the same as the message any other person present receives. Minor differences may be due to acoustic circumstances, familiarity with the language and perhaps peculiar terminology. Otherwise the message that p is uniform within the audience.

Second, B forms the opinion that A has a thought that p. B may add that A only considers whether or not p, or can add another mode. B reconstructs a part of his niche for A. B sometimes will use his previous knowledge of A's opinions to draw new conclusions from the message that p, conclusions within the niche for A. B may claim that A knows this and that, but also that A considers, or dislikes other thoughts, is amused by still a different set of statements and, if A is a judge, that he rules that p.

Third, B tries to see what is for himself the content of the message, i.e., what consequences he can draw from the message that p. For that purpose he uses all the arsenal of his experience, he can use any sentence he thinks is true. But in practice he often jumps to conclusions faster than he realizes what additional premises he is using. This arsenal of accepted assumptions is in a niche for B which a researcher may try partly to discover.

It might be observed that this paper starts the study of communication – and thereby of a large domain of linguistic facts – by the study of *parole*, or, in another theoretical frame, by the study of performance, or, in a still different terminology, by pragmatics. Such observation would be, in essence, correct. But this should not be construed as my rejection of other, more abstract problems, of *language*, of competence, of restrictions on grammars, of ignoring the roles of the speaker and the hearer. Such abstracting from the details of *parole* allows us to achieve powerful and insightful generalizations. My choice here (and not necessarily in other writings) to start by examining conversations has several motives. First, I am trying to sketch a logic or a grammar for historical studies. I am not concerned here with the history of language but with a history of people, concrete people in real situations, when they speak or write to one another. Second, facts about conversations are of importance for the study of oral discourses rather than written formal papers. Omitting repetitions, leaving out many words in a sen-

tence, using "scrap" sentences, giving short answers and corrections, using enthymemes, i.e., not saying what is well known, call for systematic study. Furthermore, it is fascinating to observe how the various abstract grammars are related to the pragmatics of speech. Speech constitutes the original data of any grammar. How does a grammar deal with those data? Semantics often assigns abstract structures as values to utterances. This is an important and fruitful procedure. But in the real world individual people understand the utterances and they understand them in a variety of ways, creating mutual misunderstandings. In the last hundred years logic was dominated by analysis of the language of science, mainly mathematics. The natural language is much richer and a reduction of its logic to the logic of science does not succeed without a curtailing or, to use Quine's phrase, regimentation of the natural talk. Logic prescribes rules for correct proofs in science. But it is artificial to say that people apply any rules when they speak, understand and draw conclusions. To know a language is to speak it rather than to follow rules. Grammar is a description, maybe an explanation, of speech, not a precondition for speaking. Similarly, quantum mechanics is an explanation, not the reason, of physical events. Next, I am taking as bare data of linguistics not only sentences and people's judgement of sentencehood, and not only people's judgement of coherence of texts, but also the facts that people relate sentences to each other in a systematic way, draw consequences, that they deny, correct, reply. Augmenting the empirical base of research allows simpler theoretical mechanisms. Instead of predicting the consequences drawn by people from a sentence, I take them as given. Finally, no doubt, I am motivated by methodological behaviorism. Language is external, as Quine puts it. This does not mean that speech does not show how people think, how they understand. The bare data are external and consist mainly of people uttering forms and relating them to other forms. The processes of thinking and understanding will certainly prove to be electrical impulses and chemical changes. For the moment, we call them mental life of people. Just as a physicist describes the structure of an atom from photographs of paths of electrons, a linguist reconstructs or rather constructs the internal processes of thinking from speech. But as long as we are far from an electro-chemical description of thought, thinking is a net of utterances, not necessarily only of actually used utterances. This may be called an enlightened behaviorism.

The entire sentence of form (1) is in the language we use. We call it

U-language.[1] The form in which the event that p is rendered is a U-form, usually a sentence, sometimes a sentence suitably adjusted to the way of saying. *(Peter relies on the fact that she is kind, Peter relies on her being kind, Peter relies on her kindness).* The form α is not necessarily in U. It may be in a language foreign but understandable to the listener (perhaps via an interpreter). It must be a sentence, or a form which stands for a sentence or for a longer utterance which has a sentential import. The sentence 'p' in (1) is a translation of α into U. The person A is taken at a historical moment. The person C reports to us in his U-utterances, perhaps through intermediaries who repeat or translate (the more the worse). The audience and the reporter may have been at different places or moments than A. For instance, they may have heard A from tape on the radio.

When we say that α is in the language of the speaker and that the event that p is rendered in the U language, we do not really speak about languages. When we say that the Latin sentence by Cicero, *Nulla ars imitari sollertiam naturæ potest* translates into the English sentence *No art can imitate the ingenuity of nature*, we use *Latin* and *English* as adjectives. To judge that a form is in Latin or English is much less than to use a highly abstract concept of a language.

When a historian of philosophy reports the opinion of Cicero, he forms a semantic niche for Cicero and places in it that no art can possibly imitate the cleverness of nature. This is how Cicero was understood by his contemporaries and by us. Now, a certain artist may think that his art can imitate the inventive skill of nature. Do we have a contradiction? Certainly not. One opinion is in the niche for Cicero, the other in the niche for the conceited artist. We must keep the two niches apart. We usually do. When we do not, considerable historical, semantic and philosophical complications arise. A prohibition must be stated: do not reason within any niche using premises not in it. You can compare the statement in a niche with statements not in it, you can criticize a part of a niche. But then you are outside it. In the niche of one person for another are the opinions of the first person about what the opinions of the other are and not the opinions of the first person about the opinions of the other.

In difficult matters symbolism helps to lessen the confusion. Let us keep record of who said what to whom and who reported it. I will sometimes use "mode" instead of "said in a mode". Between a pair of slanted bars "/" and "\" will appear an element of a niche. John said

that /today is Tuesday$\backslash$. More precisely, I tell you that /John said that /today is Tuesday$\backslash\backslash$. The schema (1) will be reformulated as (2).

(2) C in a report mode $/A$, to B, mode $/p\backslash\backslash$

If A, using one form or another said to B that $/p\backslash$, then and only then from A's utterance B got the message that $/p\backslash$, or understood that $/p\backslash$. We can now say that A uttered α from which B understood that $/p\backslash$. This formulation puts a proper focus on B. Instead of (2) we may use the more specific (3).

(3) There was an encounter between two people, A and B, from which C understood that $/A$ uttered α from which B understood that $/p\backslash\backslash$

The understanding by the reporter C comes not as a transfer of information from another speaker, but from observation of an event, namely the encounter of A with B. There might have been some record from which C_1 understood that $/C$ from an event understood that $/A$ uttered α from which B understood that $/p\backslash\backslash\backslash$. Similarly, there was perhaps some record from which C_2 understood that $/C_1$ from another record understood that $/C$ from an event understood that $/A$ uttered α from which B understood that $/p\backslash\backslash\backslash\backslash$. And so on.

If C_1 has confidence in the record and in the understanding of C, then C_1 may decide to inherit the understanding of C, and then C_1 understands that $/A$ uttered α from which B understood that $/p\backslash\backslash$. In C_1's niche for C is that $/A$ uttered α from which B understood that $/p\backslash\backslash$. In C_1's niche for C is that /in C's niche for B is that $/p\backslash\backslash$. If C_n is our historian and if he has enough confidence in he records and in the "apostolic tradition" of previous C-s, then he understands that $/A$ uttered α from which B understood that $/p\backslash\backslash$. From this, the historian concludes that $/p\backslash$ is in B's niche for A. (Here, as in some other places, for simplicity I omitted the modes.)

To say comes from a long list of verbs that take an entire sentence as their complement. An important property of those verbs is that any sentence can be their complement. (The only restrictions are on the iteration of some of those verbs, e.g. **I think that I doubt.*). Such verbs are *say*, *assert*, *write*, *suggest*, *propose*, *understand*, *report*, *imply*, *think*, *doubt*, *deny*, *lie*, *command*, *omit*, *state*, *whisper*, *rule* and hundreds other verbs. They are called container verbs.[2] Most of them require the form *that* in front of the contained sentence. Some verbs allow, and some require, other variants of sentences: *I order that you should write the*

check, I order that you write the check, I order you to write the check. Also *Father asked you whether she cooked the dinner, Father asked you who cooked the dinner, Father asked you what she cooked.* In the symbolism of (2), *(Father, of you) Past (ask) /whether ...\, /who ...\, /what ...*, ... It will be useful to consider a direct quotation as a particular case of a container verb. In such a case we write *say verbatim.* For instance, Suetonius reports /that Titus, after a day without a good deed used to say verbatim /"diem perdidi"\. The verb *report* in (1) and (2) is a container verb and can be replaced by *say.*

When we have no information, or interest, we do not say to whom the utterance was directed. Certainly, we do want to know who is the addressee when the statement is of the sort of *You are wrong.* Then, we cannot report what was said by a word-by-word translation or repetition of *You are wrong.* If the addressee is called *Felice*, then the reporter will say that /Felice was wrong\. The resolution of referentials between the form used and the message, is not always as simple as in this case, and it may lead, and has led, to historical mistakes.[3] When the reporter C is omitted, the historian himself takes the responsibility for the report. In history books, the sources are given in footnotes, if at all. A careful historian evaluates and compares the sources, but he seldom shows the tedious part of his laboratory work to the public. When you invite guests for dinner, you do not show them the kitchen, as Leonard Bloomfield used to say about the details of his work. In the end, we only have the niche and who it is for. When the speaker is omitted, confusion may result. That may be acceptable in some cases. We receive many messages, especially in written texts, of which we ignore the author. Such are mathematics textbooks or the current news. Only historians of mathematics want to know who wrote the book. It would be funny to say, a week after the event, that the New York Times reported that Gorbachov resigned as president.

The verb *say*, as used in (1), is intended as a primitive, or as a fundamental, term of semantics. It connects persons, their speech forms and the mode in which they were uttered with the message they convey. *Say* is a term of strong semantics. Weak semantics may study only the relation of synonymy and other relations between utterances themselves, e.g. complementary distribution, or hyponymy. Strong semantics relates speech to the "world", i.e., to situations and events. I hope that *say* will suffice to build a strong semantic theory. This hope is in harmony with the view that speech is a transfer of information. Information is

given in utterances which are like sentences rather than like separate words. There is a long tradition of sententialism in semantics, from Bentham, Frege, Russell to Quine.[4] With the advent of the concept of forcing which links situations to sentences, the use of *to say that* as an important semantic term has gained powerful allies.

By gathering reports of additional reporters, niches can be augmented.

(4) if $C_1/A/p\backslash\backslash$ and $C_2/A/q\backslash\backslash$, then $(C_1$ and $C_2)/A/p, q\backslash\backslash$

In (1) *reported* can be considered as a short for *said reporting*. Similarly, many other verbs can be interpreted as modifying the verb *to say* which occurs in front of them, although in zero form, i.e., deleted. It can be deleted because it adds nothing to the message. *Exclaimed = said exclaiming*, *objected = said objecting*, *feared = said with fear*, *omitted = said b omitting a*, (as one cannot omit without saying something), *denied that p = said that Neg (p)*, (where *Neg (p)* is the result of the negation transformation applied to the affirmative 'p'), *wrote = said in writing*, *repeated = said again*, *copied q = repeated q in the mode in which q was said.* I do not attempt to reduce all container verbs to *say*, event though it is quite clear that *say* is central for strong semantics. The verbs *say*, *tell* and *speak* are in nearly complementary distributions. *Speak* does not take *that* with a sentence. *Tell* takes *her* or some other dative. *Say* does not. The verb *read* is instructive by the ambiguities in the scope of niches. If the reporter tells us that A read that p, it can be taken in, at least, four different ways.

read$_1$ – C said$/A$ said to B/A read α and the author of α said in writing $/p\backslash\backslash\backslash$
read$_2$ – C said$/A$ said to $B/$the author of α said in writing $/p\backslash\backslash\backslash$
read$_3$ – C said$/A$ read α and the author of α said in writing $/p\backslash\backslash$
read$_4$ – C said$/A$ said to B/A read $\alpha\backslash$ and the author of α said in writing that $/p\backslash\backslash$

In *read*$_1$ and *read*$_3$, *read* without a subscript refers to the asemantic act of going through the text. In *read*$_2$, A's knowledge of α may be by intermediaries, like our knowledge of the writings of Democritus. In *read*$_3$, A does not attest to his own reading of α. Only C said that. In *read*$_4$ the readings of α by A and by C may differ.

Dialogue is the most natural way of talking, where to what somebody has said somebody else verbally reacts. The second person replies, or retorts, opposes, answers a question, corrects, add something, inquires,

appreciates, or sneers. Soliloquies are artificial, homilies and lectures are exceptional and are addressed as a rule to a larger audience. Successive commands are unbearable. A dialogue takes place when the person addressed by A is the next speaker B who in turn addresses A. When C reports the dialogue, we have the schema

(5) C mode$_1/(A, b, \alpha)$ mode$_2/p\backslash$ and contiguously (B, A, β) mode$_3/q\backslash\backslash$

A reporter tells us that a person said something to another person and then the roles changed and the second person said something to the first. The syntactic form β is normally connected to the form α. A short answer β is a typical example; all sorts of pro-forms and zeroings link it to some parts of α. When A takes his turn again, he may connect his new utterance both with α and with β. The dialogue builds up its own grammar and its own semantics that a third person who has not heard the beginning may find it difficult to grasp correctly, if at all. A dialogue is a joint performance which is a test of the mutual understanding of the participants. Sometimes, even if A can correctly repeat what B said, in his replies he may attribute to B statements not made by B and which B does not agree with. It is a case of misunderstanding. Misunderstandings pervade our life. The *Comedy of Errors* is also a comedy of misunderstandings. There is a difference in understanding when two people draw different conclusions from an utterance. We then say that although both persons know what was said, each assigns to it a different content. It is impossible to be certain that people understand a form in the same way. The best we can hope for is sometimes to find out when they do not, and why. When the speaker confronts some of the consequences drawn by his interlocutors from his utterance, he may be surprised and may protest. This happens in dialogues. The misunderstanding can manifest itself only when the speaker converses with others and hears how they have understood him. In a monologue or in a homily no sign of understanding by the audience is evident. Nor, for that matter, by the speaker.

We speak, listen, and we also conclude. We reason and draw conclusions from utterances, from sentences and from functions. From the fact that there is an office of the president of France we conclude that France is not a monarchy. He rested, therefore he was tired. The same holds for functions. If x^2 is an even number, so is x, and for some integer y, $x = 2y$. Consequences of a single sentence are usually trifling. They

do not say much more than the sentence itself. Ordinarily, we draw consequences from several sentences taken together, often from a sentence with some assumptions. The term "content" for the set of consequences of a sentence was introduced by Frege ("Inhalt"). The assumptions come from our previous knowledge, the preceding utterance or from the environment shared by the speaker and the listeners. The majority of assumptions are left unstated, because one supposes – rightly or not – that the listener makes the same assumptions as the speaker, or at least is ready to accept similar assumptions. According to Frege,

(F) With assumptions X, the content of α is $Cn(\{\alpha\} \cup X)$[5]

It will be called *Fct*. This formula needs a correction, because – as Carnap noted – according to (F) all the consequences of X, even those having little to do with α, are part of the content of α. Thus, Carnap introduced

(C) With assumptions X, the content of α is $Cn(\{\alpha\} \cup X) - Cn(X)$.

It will be called *Cct*. Irena Bellert observed that (C) is inadequate for the cases where α is among the assumptions X, because then the content of α is nil. (C) is rather a representation of what we learned from α on top of what is already known from X. Bellert proposed

(B) With assumptions X, the content of α is $(Cn(\{\alpha\} \cup X) - Cn(X)) \cup Cn(\{\alpha\})$[6]

It will be called *Bct*.

To be useful, the concept of consequence from (F), (C) and (B) must cover the cases of consequence in natural discourse. Let us consider two elementary examples.

(a) Every teacher is underpaid; Ellen is a teacher. Therefore, Ellen is underpaid.

(b) We crossed an iron bridge. Therefore, we crossed a bridge. It was of iron.

More generally

(c) Every P_1 is P_2; N is $P_1 \Rightarrow N$ is P_2

(d) $F(N_1N_2) \Rightarrow F(N_2)$. Ref($N_2$) Tense (be) Case (N_1)

where Ref(N_2) is a referential to N_2, in this case an anaphora to *bridge*. (a) and (b) are records of an actual acceptance of a consequence on the basis of its premis(es). (c) and (d) are generalized descriptions by

linguists of those and many similar acceptances. The differences between premises and consequences are described in terms of movement, addition or deletion of words. The general statements are formulated with the aid of variables and grammatical constants only, such as *and*, *or*, *every*, *whether*, *he*, *which*, *what*, grammatical suffixes, and certain prepositions (which indicate the so-called dependent cases). Grammatical constants occur in almost all kinds of speech, in the press, in fiction, in chemistry, in judiciary proceedings. Exceptions are those texts that essentially depart from everyday language, as do the codes of law, mathematics (in formulae, not in commentary), the formulae of molecular chemistry and, of course, logic. The structures assigned to premises and to conclusions are the result of our descriptions of them, not the other way around. There is no justification for imputing to the speaker's mind, or brain, or whatever, the mechanisms used by the linguists or by the logicians in the descriptions. The linguistic behavior is the assent of listeners to some statements when other statements are clearly assumed. We do not observe in the listeners their reasoning, application or rules, perceiving of sentences as structured. We observe only a conditional assent which we call drawing of consequences or conclusions. In (a) one makes the conclusion that Ellen is underpaid without going through substitution of "Ellen" for "Every teacher". Few people are aware of the relation between a quantifier and a name and those who are presumably had a course in logic. Instead of *Every teacher is underpaid* it could be *Teachers are underpaid* with the same conclusion. Even *Ellen is a teacher* alone suffices; the other premise, being well known, can be omitted.

Logic and grammar try to explicate the concept of consequence. In logic there are two main ways of defining consequence, each with many variants. According to one, the semantic theory, consequence is a truth preserving relation. In any domain in which premises are true the consequence is true as well. According to the second, the syntactic way, there are rules of inference that refer to the shapes of inscriptions only and not to their semantic properties (like truth). Modus Ponens, rules of substitution, of distributing quantifier over implication are examples of such rules. A sequence of applications of those rules to the premises leads to a consequence. Both ways are illuminating. In recent years, logicians in their work use extensively results and methods of other disciplines: algebra, model theory, probability. Logic is being profitably contaminated, or inseminated. It seems pointless

to insist on its old boundaries. More constants are analyzed by the extended methods, e.g. *many*. But the gap between today's logic and its applicability to natural discourse remains considerable. In some formulations of grammar, statements of transformations seem close to the syntactic definition of consequence. They refer to grammatical constants just as logical rules refer to logical constants, as in (c) and (d). The transformational grammars did not go far enough in stating rules relating a given sentence to many alternative sets of premises. Perhaps this fact was the result of the requirement, put on those grammars in the early stages of the transformational theory, that a sentence have only one derivation. Two ways of generating a single sentence would indicate a structural ambiguity of the sentence, or redundancy of the grammar. Of course, the goals of those theories were different than a study of consequences. The relation between reasoning performed in a natural discourse and logic or grammar is like the relation between walking and mechanics and anatomy. We walk. Mechanics and anatomy try to show how it happens that we walk. No knowledge of mechanics or anatomy is needed for walking, and if those sciences do not explicate all our walking ability, they must improve. Logic and grammar have far to go to describe in detail our reasoning. *Nulla scientia imitari sollertiam naturæ potest.*

In order to clarify the content of α for a person A with assumptions X, we must establish which members of X are used by A when he or she draws a conclusion β from them and from α. Not necessarily the entire set X is used. Conclusions drawn by A will be symbolized as '$Cn{<}A{>}$'.

(6) α is $Cn{<}A{>}(X)$ if and only if the person A agrees that α is a consequence of the set X.

We do not have to add in (6) that X are assumptions. For, the set X is not a set of sentences A holds true, but only sentences A at a moment is confronted with, considers, processes without acceptance, and if A agrees that α is a consequence of X, then he is confronted with X, no matter what is his other attitude towards X. It may be advisable to look for a sharper concept of the content than *Fct*, *Cct* or *Bct*. For that purpose let us turn to a more familiar ground of a scientist who asks himself whether α is a consequence of statements which he accepts. If his answer is positive, then he will examine which of his previous tenets played the role of premises in the deduction of α. After that –

and this is crucial for the present discussion of content – he examines whether the used premises were independent of each other, whether all were necessary to deduce α. If one of them is deducible from the remaining, he will eliminate it from the list of assumptions on which he based α. Thus he forms, as it were, an axiomatization for α where the axioms are independent each from the others and all the axioms are among his previously accepted statements. This set will be called *an arrangement of relevant premises for* α, in abbreviation *Arp*. A characteristic property of an *Arp* for α is that α is derivable from it, but not from any of its proper parts.

(7) X is an $Arp{<}A{>}(\alpha) \equiv (\alpha$ is $Cn{<}A{>}(X)$ and if Y is a proper part of X, then not $(\alpha$ is $Cn{<}A{>}(Y)))$

More symbolically,

$$\begin{aligned} X \;&\in\; Arp{<}A{>}(\alpha) \\ \equiv\;& \Big(\alpha \in Cn{<}A{>}(X) \wedge \bigwedge Y[Y \in Pp(X) \\ \supset\;& \sim(\alpha \in Cn{<}A{>}(Y))]\Big)\,. \end{aligned}$$

Here, '$\in$' is used as by Leśniewski or as 'is a'. Sometimes, when it is obvious who is referred to, I will omit the relativization to a person. There may be many different *Arp*s for α within available assumptions of the scientist, just as there may be different axiomatizations of an axiomatizable set of statements. Now, the term *content* (now marked simply by *Ct*) will be used in such a way that a statement is in the content of its *Arp*.

(8) $\alpha \in Ct{<}A{>}(X) \equiv X \in Arp{<}A{>}(\alpha)$

Formula (8) resigns from assigning a content to a single statement. Rather, it assigns it to an arrangement of relevant premises taken *en bloc*, premises for an element of the content. The content of the *Arp* is composed of all its consequences which require all elements of the *Arp* as its premises. If a consequence of an *Arp* for α is obtainable from only a part of the *Arp*, then it is in the content of that part and not of the entire *Arp* for α. None of the sets of statements here should be identical with the set of all statements (whatever it may be) and therefore contradictory in this sense.

Now we can modify Carnap's and Bellert's contents to restrict them to relevant assumptions made by a person. Then we can compare them with the concept of content as given in (8). Everything in it is in the

Bellert's content:

(9) $Ct{<}A{>}(X \cup \{\alpha\}) \subset Bct{<}A{>}(X, \alpha)$

(10) Suppose that $\beta \in Ct{<}A{>}(X \cup \{\alpha\})$. If $\sim (\alpha \in X)$, then $\beta \in Cct{<}A{>}(X, \alpha)$. If $\alpha \in X$, then $\beta \in {<}A{>}(X), X = \{\alpha\}$, and $Cct{<}A{>}(X, \alpha) = \emptyset$.

This makes good sense. If a statement is among somebody's assumptions, he does not have to use any other assumption to derive it. Indeed, he does not have to derive it. To take a simple example, suppose I told a friend that I love to travel by train. He concluded that I am conservative. I protested because I do not consider myself conservative. As a substantiation of the accusation, the friend stated that trains are passé. I tried to show him that trains are improving and working well in France and Japan. Finally he admitted that his premise was incorrect and admitted me to the ranks of progressive men. But even if my friend had not been persuaded by my augmentation, I could have asked him whether he would continue to assert that I was a conservative even if the trains were not passé. If his answer were *you would not be*, trains being passé would have been his premise for holding me conservative. In this case I attribute to my friend a suspension of his opinion. When he says that in case the trains were not passé, I would not be a conservative, he performs what Coleridge called a "willing suspension of disbelief". He makes a little, provisional niche. We often reason from contrary to fact, or contrary to persuasion, premises. Man is a niche playing animal.

Most of the examples in this paper have an assertive mode, which with *say* becomes *say assertively*. But there are messages which hearers receive without the assertive power. The reporter said that /Peter doubts that /aspirin is a good drug\\. Therefore Peter doubts also all assertions to which form *aspirin is a good drug* is a relevant premise. He also doubts all the assumptions which for him are relevant premises to *aspirin is a good drug*. We are facing a bookkeeping choice. Either we form for Peter several niches, each for a different mode: a niche for assertions, another for doubts, others for plans, for fears, etc. and permit operations on these sets, e.g., inclusion of a stronger set into a weaker one, the set of Peter's knowledge into the set of his doubts, but not vice versa. Or else, we keep a single niche for Peter and impose within it some modal logic, but much richer than the popular ones. In either case the number of modes will be large. Nearly every container verb leads to a different

mode. For the rest of this paper I will follow the second strategy. For Peter at a given situation there will be only one niche by one author, unless he hesitates and gives two interpretations of Peter. Also, it may be useful to form a niche for Peter as a chemist and a separate niche for him as a family man.

As the discussion now stands, there is a technical reason that we cannot write in the niche for B the content of α for B as defined in (8) for A. The content of a set of forms in (8) is a set of sentences, whereas the elements of a niche should be events. In order to overcome this difficulty we may reinterpret the concept of consequence as holding not between sets of sentences but between sets of events. It calls for replacing 'α', 'β' and 'γ' by 'p', 'q' and 'r' respectively, or sentences by their semantic values. Generally, for any set S of utterances, $[|S|]$ is the set of semantic values corresponding to the elements of S. If B hears α with understanding, we can add $[|Ct{<}B{>}(X \cup \{\alpha\}|]$ to his niche. Equivalently, we could define a relation of consequence not for sentences but for events. This, however, is less familiar. Accordingly, instead of (5) we may describe a dialogue

(11) C mode$_1$ $/(A, B, \alpha)$ mode$_2$ $/[|Ct{<}B{>}(Y \cup \{\alpha\})|]\backslash$ and contiguously (B, A, β) mode$_3$ $/[|Ct{<}A{>}(X \cup \{\beta\})|]\backslash$

In other words, C reported in a certain way that B was told in some mode by A, who used α, that p ($p = [|\alpha|]$) and, recalling some events Y, B drew conclusions; in reply B used β and in a mode told A that q ($q = [|\beta|]$) and using some of his own recollection X, A drew conclusions. When B hears α, he may form a suspended content for it. If he is surprized that p, he may compare it with his assumptions relevant to α. Normally we want to see where a proposal would lead before we accept it. An utterance has more content in a dialogue than when it stands alone. To see that let us return to the semantics of (3). When C reports that $/B$ understood the utterance α by A that $/p\backslash\backslash$, C bases his assertion on B's behavior. B could have been following advice of A. C may have heard B's report of what A said. Normally, both C and B know that A spoke. But what A said does not contain that information. Often the message says nothing about its origin. Similarly, when someone speaks English with a Boston accent, he does not say that his accent is Bostonian or even that he speaks English. Of course, the listeners notice these facts. If someone utters *17 is a prime number* we cannot take him as saying that he knows that 17 is a prime number.

We are told something about the number but we notice that he knows it. There is an important distinction between what we are told and what we notice. The distinction may seem pedantic, but without it semantics disintegrates. We notice many circumstances surrounding the message.

In a dialogue the assumptions of the participants are changing, at least as the result of what they hear from each other. Let us call a situation an event in which the assumptions of participants do not change. Let the person A in the situation s make assumptions Z. Now, we are ready for a definition of a niche.

(DfNiche) $[|\alpha|]$ is in a niche which B formed for A in $s \equiv B$ said affirmatively $/ \vee X[X \subset Z \text{ and } X \in Arp\langle A \text{ in } s\rangle(\alpha)] \backslash$

Less formally, the semantic value of the sentence α is in B's niche for A just when B claims that among A's assumptions there are statements being independent and necessary axioms for α, and that A would accept that α follows from them. What are exactly the assumptions Z may be hard to decide. But its subclass X must be relatively sharp in order to have a relatively sharp niche. If the subject matter is stable enough, the situation may last a long time and in that topic a niche for a person may not change either.

In some cases the message itself gives the information as to who is speaking. An utterance by Felice may start by *Felice is speaking*, as is usual in telephone conversations. She may give us some information about her thoughts. She may say *My apartment is in Concord. I recall that the train was late.* Then the listener is informed about her recollections. He can reason from the premise that the train was late and form a small niche about Felice's recollections.

The situation changes radically in a dialogue, where Felice can, and usually does, receive a reaction to her statement that her apartment is in Concord. Marco can reply *Oh, I was wrong. I thought it was in Boston.* If she is told that, i.e., if she understands that Marco thought that her apartment was not in Concord but in Boston, she assigns to Marco's utterance the content which indirectly includes her first utterance. Now a semantic niche for Felice is constructed by observers, especially by Marco. The observer can see that she is not only the producer of a dispatch but also a receiver of a message about her dispatch. Therefore her dispatch is now a part of a semantic niche for herself. This is a typical fact about dialogues. If the recipient assigns content to her utterance, then in a dialogue Felice has a chance to be a recipient of her own

utterance, to give content to her own previous utterance, to show how she understands it and perhaps to correct other people's understandings of it, maybe to realize what she was saying. Not just speaking but being spoken to makes us members of society, participants in a conversation.

When we read an old text we may do it in one of two ways. We may try to read it historically, as a contemporary of the author would have read it, using only those assumptions we think were made by the public to whom it was addressed. We thus place the text in the intellectual territory of the author, we seek what he was reacting to, what was novel and of interest in his text. We try to recreate the content with which a contemporary to the author would have read it. The other way of reading the text is to ask what a reader can learn from it today, what ideas he can derive from it, what it says to me. I then receive the text with my own assumptions, I give to it our modern content. This is what is often called 'hermeneutics'. As an example of the second way of reading may serve Łukasiewicz's work on Aristotle's and Stoic logic. Łukasiewicz's interpretation of Aristotle was criticized by John Corcoran and of Stoics by Michael Frede. Either way of reading is justifiable and useful. They only need to be kept apart. We can brush our teeth and brush our shoes, but we should not confuse the brushes. These ways of reading are radical cases. Most often, we read mixing the two types. Even Michael Frede is not reading Stoics with the attitude of hellenic intellectuals but with his niche for them.

Sometimes people make untrue statements and sometimes, but less frequently, they reason incorrectly. It seems that logical correctness of substantiation is ingrained in the culture more deeply than is truth. Certainly, not everyone is a master of proof, but bunglers are few, and even they seldom commit errors of reasoning. Usually, the apparent error of reasoning turns out to be the use of a false premise. People differ in opinions, but they do not differ essentially in logic. All the time we reason from premises the truth of which we do not know, or we do know that they are false, but we wish to learn where such premises lead. We also reason from premises that are sentential functions; such a reasoning follows a course similar to that of a reasoning from actual sentences. What is different, is the care needed not to confuse free variables. In every day speech, the same kind of care is required. In the text *John entered the kitchen. With a kiss Mary killed a mosquito* there is a confusion as to the complement of *kiss*. The patterns of reasoning seem to be a part of language structure. While errors in reasoning put one's

linguistic competence in question, just making false statements need not. Someone making a false statement may be thought incompetent in the field under consideration, but not in his own language. It is a common saying that the masters of rhetoric are expert liars. For formalized languages, grammar is the same as logic. For natural languages it is not much different. Of course, the logic of mathematical languages must not be confused with the logic of everyday English. I doubt whether in a sophisticated conversation or monologue Peirce's Law *CCCpqpp* would be used. It is even doubtful whether the word-for-word translation of that law into English is English: *If if if p then q then p then p.* A formula in which three implications are successively nested, i.e., a formula with *CCC* cannot be translated and it would be hard to interpret it in English. There is no way to assign to it a suitable English intonation. The iteration of logical functors is limited in natural languages. The functor *not* does not iterate in English, nor does it in Polish. *Peter is not not a teacher* is not a fluent English sentence. The negation has a binary, flip-flop effect: the negation of an affirmative sentence is its denial, the negation of the denial is the affirmative sentence.[7]

Naturally, people's assumptions are to some extent similar, that is, their experiences, opinions, knowledge, prejudices have much in common. Without this we could not survive. Perhaps we could try to determine what assumptions are shared within a social group. The task would not yield precise results. Such research could begin with noticing that the sentences judged the most banal are not said. *Tuesday follows Monday*, *Mothers are women*, *Children are younger then their grandparents* are sentence nearly never said and are assented to by everybody, perhaps with a faint smile. A large part of the common knowledge remains tacit. Discovering hidden, enthymematic assumptions of a text is a titillating task for a linguist, who noticed that an author was engaged in a seemingly incorrect reasoning. It can often be shown that some premises are missing. And it can be frequently determined in what area they would lie. Sometimes such premises can be found elsewhere in the text, or in another text on similar topics.[8]

We have agreed that the listener receives the message that *p* and reacts to it with assumptions in his present niche, which we or the reporter, constructed for him. The problem remains whether there is a unique message from a dispatch. In the style of this paper, the supposition of uniqueness of the semantic value of an utterance may be formulated as

(12) If C said $/(A, B, \alpha)$ said $/p\backslash\backslash$ and C said (A, B, α) said $/q\backslash\backslash$, then if p, then q

In effect, (12) asserts the equivalence of messages originating from one dispatch. (12) is a distant derivative of Frege's law "courses of value" (Werthverlauf):

(13) If a is the class of objects which have the property f and a is the class of objects which have the property g, then for every x, if $f(x)$, then $g(x)$.

In the spirit of Russell's theory of types, or semantical categories of Leśniewski, f and g in (13) can be functions of any number of arguments. If they are functions of zero arguments, f and g are sentences. (Neither Frege, nor Russell explicitly considered sentences as zero argument functions.) Giving a linguistic interpretation to (13), and abstracting from speaker and hearer,

(14) If ((α says that p) & (α says that q) & p), then q

Frege also accepted that for every property there are objects that have that property, which in the linguistic interpretation means that every event is describable by a sentence. That is certainly false, at least if it requires that for every event there be a different sentence. For, there is at least a continuum of events and at most denumerably many, perhaps finitely many, sentences.[9] (Of course, in Frege there was no such limitation.) The formula (14) is doubtful. From previous comments it would seem that the relation between the dispatch and the received message may not be that simple. Sometimes it is possible that the listener is missing some of the nuances of the speaker's performance or has lost a part of it. Then it would be more suitable to say that, using α, A said to B that at least p. Now, suppose that someone said to his two companions "We have to do something about this horrible window". One of the companions understood it as saying that the window is dirty and that they have to wash it, the other that the window is drafty and needs insulation. When it became clear that the problem was with the dirt, one companion wanted to clean only the pane, the other also the frame; one thought that it should be done immediately, the other within a few days. Whatever is said can be further specified. To take another example: the speaker said "The river Dnieper is more than twice as long as the Vistula". Did he say by how much the Dnieper's length is greater than twice the length of the Vistula? Is the Dnieper three times as long? No. The habits of speaking about rivers require that the Dnieper be

less than three times Vistula's length, maybe less than two and a half. On the other end, if the Dnieper were a hundred meters longer than the Vistula, the habit would require saying that they are of equal length. Is there a situation which is the most suitable semantic value for the speaker's utterance? One could propose that the situation in which the Dnieper is between 2.1 and 2.45 times longer than the Vistula is what the utterance said. But further, or different, approximations are always possible. Whatever I say you can paraphrase in more than one way. And it may be hard, or impossible, to decide which one is the closer paraphrase. Maybe there is no closest paraphrase, except repetition word by word preserving intonation.

To elucidate this problem, recall that the algebraists talk about the situations, or events, within a domain as partially ordered by the relation of being more specific. For instance, the Dnieper being more than twice as long as the Vistula is more specific than the Dnieper being longer than the Vistula. We write $p \leq q$ if and only if q is at least as specific as p. In such cases $q \supset p$. The existence of the maximal, the most specific, event corresponding to a sentence may be a theoretical assumption:

(15) For any α uttered by A there is such a q that A using α said that q and such that for any p said by A using α, $p \leq q$.

The acceptance of (15) simplifies the description of linguistic transfer. But it may complicate the theory in other ways. Frege's principle (14) assumes (15). Some pairs of event cannot coexist for one reason or another. From a partially ordered domain of events, a possible world can be chosen according to two principles. First, if an event is in a possible world, so is every event less specific than it. Second, if two events cannot coexist, then at most one of them can be in that world. The probabilists call such a choice a "realization". Algebraists call it an "ideal". If a realization has the property that for every pair of events which cannot coexist one must be in the realization, it is called a maximal realization. This requirement is too strong for linguistic applications. We need events about which we can stay "undecided".[10]

A semantic theory must be tested against the difficulties (the Greeks called them *aporias*) which were encountered during the long history of logico-semantic research (that went under a variety of names: *dialectics*, *logic*, *grammar*, *semantics*). To test it, short comments about six of them will help.

A1. The Morning Star and the Evening Star are the same planet. As

is well known, this prompted Frege to develop his theory of sense and reference. In the view presented here the form *the Morning Star*, as it stands, is not a message, unless it is taken as a nominalization of *A planet sometimes shines alone in the morning* (*star* is replaced by *planet*; the reconstructible verb *shines* is supplied). Similarly, *the Evening Star* is a nominalization of *A planet sometimes shines alone in the evening*. The semantic values of these statements are two different events. Astronomy teaches us that the planet which shines in the morning at other seasons shines in the evening. In other words, the function λx [x shines alone in the morning] and the function λy [y shines alone in the evening] can be combined, by the operation of identification of variables, into λx [x shines alone in the morning at some seasons and x shines alone in the evening at other seasons]. The combined function is applied to a celestial body, namely Venus: λx [x shines alone in the morning and x shines alone in the evening] (Venus). Here Venus is another function λx [x is the second planet from the Sun]. In this treatment of the puzzle two points are essential; taking *Morning Star* is a nominalization of a sentence and the identification of variables of two functions.

A2. Kepler knew that (a) all planets orbit in ellipses. But (b) Uranus is a planet. Hence (c) Kepler knew that Uranus orbits in an ellipse. But Uranus was discovered by Herschel in 1781, a hundred and forty years after Kepler's death. This is a problem of maintaining a proper niche for Kepler. In that niche there is (a), but not (b). A reasoning within a niche should not use premises from outside. If somebody says that Kepler knew that every planet which may be discovered in the future orbits in an ellipse, I will reply that in our use of the term *Uranus* there are several statements that we accept as true, such as *Uranus year lasts about 84 Earth years*, which Kepler did not know even "potentially". The content of any sentence with *Uranus* was for him quite different than for us.

A3. (a) Caesar was killed in Rome. (b) Rome is the domicile of Craxi. (c) Caesar was killed in the domicile of Craxi. We know both (a) and (b). Did Cicero know (c), because he knew (a)? No, because he did not know (b). In Cicero's niche we find (a), but not (b). The Leibnizian principle of substitutivity of identicals, x and y, holds within a niche only if $|[x = y]|$ is in the niche. Note also that the identity statement should be made by the operation of identification of variables on functions: λx [Caesar was killed in x and x is the domicile of Craxi] (Rome).

A4. Euclid knew that if $x^2 = x$, then $x = 1$. But also $0^2 = 0$. Was Euclid wrong? No; because '$O^2 = 0$' was not in his language, his domain of semantic values had no semantic value $|[\text{'}0^2 = 0\text{'}]|$.

A5. It has been proven many times over that no utterance can say that what it says is false. To state it in the style of the present paper, let us introduce a new verb *to Russell somebody*. A person, A, Russelled somebody when A said to him that whatever A was saying to him on that occasion was false. In symbols,

$$(16) \qquad AR \equiv (Vp[A \text{ said } /p\backslash] \wedge \bigwedge p[A \text{ said } /p\backslash \supset \neg p])$$

Substituting in (16) 'AR' for 'p'

$$(17) \qquad A \text{ said } /AR\backslash \supset \neg AR$$

If A said $/AR\backslash$ and A said $/p\backslash$, then by (14) $p \supset AR$,

$$(18) \qquad (A \text{ said } /AR\backslash \wedge A \text{ said } /p\backslash) \supset \neg p$$

$$(19) \qquad A \text{ said } /AR\backslash \supset AR \text{ by (18) and (16)}$$

$$(20) \qquad \neg A \text{ said } /AR\backslash \text{ by (17) and (19)}$$

A may have used the form 'I say that I Russell you', but by this he did not say that he said that he Russelled his listener. This story is often called 'the liar antinomy'. In the present formulation it is not an antinomy. It is only a limitation on what one can possibly say no matter what forms one uses. We do better to call it 'an aporia' rather than 'an antinomy'. Note that in the derivation (14) was used. Therefore, in this case, some doubts about maximal semantic values of our utterances remain. However, from the theorem that no utterance can say that what it says is not the case does not follow that the form *β says that whatever β says is not the case* is not an English sentence. Rather, it follows from the theorem that there is no possible world which contains its semantic value. Therefore, an English speaking person may use that form, but he will not say anything thereby. The situation is similar to that where somebody writes '5 divided by $0 = 5x$'. There is no possible situation corresponding to that sentence.[11]

A6. The semantic relations between sentences and words are unclear.

Several linguists take the meaning of a word for granted and advocate compositional semantics: the meaning of a sentence is composed of the meanings of its words by means of sentence structure. This was also Frege's view. A sententialist would prefer a decompositional semantics; words have a part in the process of communication just as phonemes do, but by themselves they rarely constitute messages. When they do, they are called free forms. Then, they are like short sentences. Often they are strongly connected to the context, as short answers are to questions, or *No* to a command. They are an extreme case of context dependence of a sentence. A conversation or a history book is not a list of theorems. Any utterance with a word which can occur as a free form can be paraphrased by a discourse in which that word, or its variant, is used as a sentence. Such words in Harris' theory usually are operators (that means functors). In the abstract semantics, they lead from one family of possible worlds to another, smaller or different, family of possible worlds. The difference between the two worlds constitutes a sort of meaning of that occurrence of the word.

I have just mentioned variants of a word. To illustrate, the word *never*, as used in the dialogue *You should pay taxes. – Never!* is an exclamation. In *I will never pay taxes* the word *never* is less stressed. And *never* in *Never-ending soap operas are boring* is not stressed at all. The three are pronounced quite differently. They are different forms. They have mutually exclusive distributions: wherever one occurs, the others cannot. Therefore, following the usual linguistic procedure, we may introduce a concept of a lexeme |*never*|. The three intonational patterns of *never* will be determined automatically by the context in which they occur. That much of the meaning of a word can be squeezed into sententialism easily. The words which cannot appear as utterances, the so-called bound forms, e.g., *the*, *a*, *since*, prepositions are grammatical constants mentioned before. They appear in general rules of transformations and their entire "meaning" is known in these rules.

There is yet another, more penetrating, way to study the meaning of words. Some philosophers say that the meaning of a word is in its use, or that instead of asking about its meaning we should study the actual usage of the word. Linguists do that. Shortly, the usage of a word is the totality of contexts in which it occurs. Less shortly, pairs of words often occur together (*large house*), other pairs do not (*large distance*), still other rarely and are puzzling (*large virus*). That two words do not appear together does not indicate that their co-occurrence

is completely excluded. Rather its likelihood is small. The judgement of fitness is comparative. Presumably *large distance* is less unacceptable than *large smallness*. Asking whether two words can occur one next to the other, or just in the same sentence, leads nowhere, because there are no limits at all for such co-occurrences. For every two words you can find a sentence in which they both occur and even a sentence in which they occur in juxtaposition: *On a map which is large distances can be marked.* But this example is misleading and therefore leads us to realize that between the occurrence of *large* and of *distance* there is a stop, a change of voice. And this leads further to the relativization of co-occurrence of forms to intonational patterns. A reflection of this is a different assignment of syntactic or "logical" relation between words. We say that here, and presumably nowhere is *large* a predicate, a functor of *distance* as its argument. Relevant restrictions on co-occurrences of words hold only between a functor and its arguments. We will call these restrictions 'functorial limitation'. It is a strong and falsifiable hypothesis. Sometimes the functor is in a different sentence than the argument (*Felice has a house. It is large*). It is advisable to transform an utterance we want to study for co-occurrences into an informationally equal utterance which is a chain of simple, kernel sentences. For instances, *Felice met a nice dog yesterday* can be usefully rephrased into *It was yesterday. Felice met a dog. The dog was nice.* This chain is not just a list of sentences. There are internal links between the three sentences which make the chain a paraphrase of the original (*it*, *the dog*, the tenses). Here *met* is a functor and *Felice* and *a dog* are its arguments. *Yesterday* is a functor, *Felice met a dog* is its argument. Alternatively, *yesterday* can be a functor with *met* as its argument, depending on the type of the categorial grammar. You need not form a separate sentence for *yesterday*. *Felice met* can also appear with *her obligations*. Compare the sentence *Slowly, Felice met her obligations* with *Slowly, Felice met a dog*. The first is all right, the second doubtful. It appears that *slowly* can co-occur with *obligations* and not with *dog*. But *dog* is an argument of *met*, not of *slowly*. So is *obligations*. The functorial limitation seems violated. In order to adhere to it we admit that there are two different verbs: *to meet*$_1$ and *to meet*$_2$. *To meet*$_1$ cannot appear as an argument of *slowly*. *To meet*$_2$ can. This is confirmed by the unacceptability of *Felice met a dog and her obligations*. The functorial limitation had a regulatory effect in that it forced us to change the arrangement of the data. Instead of one verb,

we have two homophonic verbs. And the independent confirmation of the result of applying the limitation partly confirms that limitation. It is expected that every word, or nearly every one, will have different co-occurrence restrictions and preferences. (*The bitch has puppies* is more acceptable than *the bitch has children*, and this is still more acceptable than *the bitch has ducklings*.[12]

In order to connect the study of co-occurrence with semantic niches, one may look at a historian who wants to establish in what way "the meaning" of a word in some circles differed from its meaning in the society at large within the same language, at the same time. He must list as many sentences with this word as he can for each group and assign the functor-argument structure to each sentence. Then, applying the principle of functorial limitation of co-occurrence to the functor and its immediate arguments, he may find that this word has different roles, i.e. that it co-occurs with some words in one group and partly with different words in the other. The procedure will be hindered by a virtual absence of negative data; he cannot ask how acceptable a given utterance is. But a clear presence of a pair in the society as a whole and a complete absence of that pair in the studied circles (or vice versa) will be significant and will serve as a piece of negative evidence. In this way he will build up two niches, one for each group.[13.14]

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NOTES

[1] The concept of U-language and the term come from Haskell Curry.

[2] The term is from Zellig Harris. Some other forms than verbs can take a complete sentence as their complement, for example, *it happens*, *it follows*, *it is true*, *is of the opinion*, *the opinion*. Any indicative sentence preceded by *that* can occur after any of them. Most, perhaps all, container verbs can take a noun phrase as complement. However, there are severe restrictions on co-occurrence of nouns with particular container verbs. *He asserts a lemma*, *suggests a change*, *thinks of you*, *thinks a thought*, *he says it*, *says grace*, *say an oath*, etc.

[3] The term *referential* covers both the anaphora and the epiphora (*When Felice was in Rome she spoke Italian*, *When she was in Rome Felice spoke Italian*). A predicate can appear as a referential (*Otto was the victor; the Emperor selected the next Pope*). Many other kinds of referentials are studied, for instance in (Harris, 1982). The term was used in a similar way by Jespersen. A semantic definition of it is in (Hiż, 1969a).

[4] A good outline of this tradition is in (Quine, 1981). Sententialism dethroned the word and

especially the noun from its old central role in semantics. With this, the concept of denotation lost its importance. It is preferable to avoid this concept altogether; there is no need for it and it leads to many formal and philosophical complications. Saying and truth are an adequate link of our speech to the world. *Truth*, or rather *true*, can serve as the primitive term of aletheic semantics as in (Hiż, 1969b). *Says that* is the primitive of a stronger theory of (Hiż, 1984), where it was used in a more classical, abstract sense than here; the first argument of *says* did not include a name of a person but solely a name of a sentence, α without A. This is a shift from a theory of an ideal language to the study of linguistic communication.

[5] Frege writes in *Begriffsschrift* (I, 3): "... the contents of two judgements may differ in two ways: either the consequences derivable from the first, when it is combined with certain other judgements, always follow also from the second, when it is combined with these same judgements, or this is not the case. ... I call that part of the content that is the same in both the conceptual content". (Frege, 1879)

[6] (Bellert, 1970). A discussion of the problems connected with various concepts of content is in (Carnap and Bar-Hillel, 1952).

[7] For this and similar reasons it would help to have the inverses of formal transformations, so that Neg^{-1} (Neg $(p)) = p$. See (Hiż, 1978).

[8] What was said during the encounter before the utterance matters for what is said now as it enters the listener's niche. But among the preceding utterances there had been incomplete sentences, sentences with an anaphora, tacit sentences. To resolve them we search for yet further assumed statements. In them there must be further enthymemes and anaphoras. We are always *in mediis rebus*. There is no first sentence.

[9] The argument for the claim that there are infinitely many (possible) sentences in a language can be as well used for the claim that there are infinitely many (possible) people. People, like sentences, are created by a recursive process and each new one is substantially different from all previous ones. Both arguments disregard physical and behavioral restrictions necessary when abstract constructions are applied.

[10] A lucid exposition of the problems is in (Łoś, 1960).

[11] I owe this comparison to Dr. Zdzisław Kowalski. The derivation of (20) follows essentially Leśniewski. See (Hiż, 1984).

[12] The method of co-occurrence may vary depending on the rudimentary grammar assumed. Even categorial grammars may differ substantially in applications. The functorial limitation is stated in (Harris, 1982). Harris uses a large spectrum of likelihood of co-occurrence and comparative judgement of acceptability of sentence in place of the more usual acceptable-or-not division. A good exposition of co-occurrence study is found in (Hoenigswald, 1965).

[13] The method of this historical study of a small group of languages is carried within weak semantics. So is the co-occurrence method. In weak semantics we do not ask what people say, only how they speak, to what extent it is acceptable to the listeners, and whether two utterances give the same information.

[14] Irena Bellert, Peter Simons and Henry Hoenigswald who have read previous versions of this paper, contributed much to my thinking and presentation. And before, there were forty years of conversations with Zellig Harris.

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JACEK JULIUSZ JADACKI

OBJECTS AND PROPERTIES

A language doing justice to the classical problems of ontology[1] must feature *i.a.* the following predicates:

(i) object,
(ii) identity (of objects), and
(iii) residing (of *properties* in an object).

1.

The residing relation is irreflexive, asymmetric and intransitive. Its domain is formed by PROPERTIES and its counterdomain by OBJECTS. We thus have for objects:

$$\bigwedge x[\bigvee y(y \text{ resides in } x) \rightarrow x \text{ is an } \textit{object}]$$

and for properties:

$$\bigwedge x[\bigvee y(x \text{ resides in } y) \rightarrow x \text{ is a } \textit{property}].$$

We have for things:

$$\bigwedge x(x \text{ is a thing } \equiv \sim x \text{ is a property}).$$

We also have:

$$\bigwedge x(x \text{ is a property} \rightarrow x \text{ is an object}).$$

As we can see, the set of objects does not coincide with the counterdomain of the residing relation. The latter is included in the former, since everything is an object and not just that in which something resides. The word "object" is said to be a transcendental term, i.e. one at once uninfinitizable and undeterminable, which may be expressed as follows:

$$\bigwedge x[x \text{ is an object } \equiv (x \text{ is identical with } x)]$$

or briefly as

$$\bigwedge x(x \text{ is an object}).$$

Jan Woleński (ed.), Philosophical Logic in Poland, 61–74.

(Conditions of identity are given below.)

Let us assume that variables:

$$x, y, z$$

range over the set of all objects, and variables:

$$P, Q, R$$

over the set of objects which are properties. We shall interpret the expression:

$$Px$$

as:

Property P resides in object x,

or:

Object x has property P.

Similarly, we shall read the expression:

$$Q(P)$$

as:

Property Q resides in property P

or:

Property P has property Q.

Assume further that:

$$\bigwedge x[x \text{ is a } \textit{superdefinite} \text{ object } \equiv \bigwedge P(Px)],$$

$$\bigwedge x[x \text{ is a } \textit{definite} \text{ object } \equiv \bigvee P(Px)],$$

$$\bigwedge x \bigwedge P(x \text{ is an object definite with regard to property } P \equiv Px),$$

$$\bigwedge P[P \text{ is a } \textit{universal} \text{ property } \equiv \bigwedge x(Px)]$$

and:

$$\bigwedge P[P \text{ is an } \textit{inherent} \text{ property } \equiv \bigvee x(Px)].$$

Let us add to this:

$$\bigwedge x(x \text{ is an } \textit{ad-definite} \text{ object } \equiv \{P|Px\} = \infty).$$

The largest ad-definite object, being one with an infinite number of properties, would be indistinguishable from a superdefinite object if the number of properties were infinite.

Let us call an indefinite object an "empty object", and a non-inherent property an "empty property". We thus have:

$$\bigwedge x[x \text{ is an empty object } \equiv \sim \bigvee P(Px)]$$

and:

$$\bigwedge P[P \text{ is an empty property } \equiv \sim \bigvee x(Px)].$$

According to the above definition of "empty object", the so-called qualityless substratum (or carrier) of properties, i.e. that which would remain of a definite object if this object were stripped of all its properties, would be precisely that empty object.

A particular kind of objects, under which we also subsume some properties, are mental or *quasi*-objects[2] such that:

$$\bigwedge x[x \text{ is a } \textit{mental} \text{ object } \rightarrow \bigvee y(y \text{ thinks about } x)].$$

Their peculiarity lies in the fact that no residing relation occurs between any *quasi*-properties and *quasi*-objects. What does occur in this case is a relation of *quasi*-residing in:

$$\bigwedge x \bigwedge P[P \textit{ quasi}\text{-resides in } x \rightarrow \bigvee y(y \text{ thinks that } Px)].$$

Mental objects (and properties) are thus in fact empty objects (properties).

In our further considerations we shall not be dealing with empty objects (or *quasi*-objects).[3] Accordingly, we shall be using the words "object" and "property" to denote objects *sensu stricto* (i.e. definite ones) and properties *sensu stricto* (i.e. inherent ones). In what follows the variables:

$$x, y, z$$

will in fact range over the set of objects *sensu stricto*, while the variables:

$$P, Q, R$$

range over the set of properties *sensu stricto*.

Let us now consider the following propositions:

(a) $\bigwedge x \bigwedge P(Px)$
All objects are superdefinite
All properties are universal

$\swarrow$ $\searrow$

(b) $\bigvee P \bigwedge x(Px)$
Some properties are universal

(c) $\bigvee x \bigwedge P(Px)$
Some objects are superdefinite

$\downarrow$ $\downarrow$

(d) $\bigwedge x \bigvee P(Px)$
All objects are definite

(e) $\bigwedge P \bigvee x(Px)$
All properties are inherent

$\searrow$ $\swarrow$

(f) $\bigvee x \bigvee P(Px)$
Some objects are definite
Some properties are inherent

Of these propositions (a) is certainly unacceptable, and (f) is certainly undeniable. If we accept that certain objects (*sensu largo*) are empty objects, and that certain properties (*sensu largo*) are empty properties, we must reject propositions (d) and (e) respectively. Proposition (b) cannot be accepted since the relation of residing is irreflexive: there is no property that would be resident in itself.

The proposition:

$$\bigvee P \bigwedge x(x \text{ is a thing } \rightarrow Px),$$

being a weaker form of (b), is disputable. This proposition is particularly important if interpreted in agreement with the previously accepted convention:

$$\bigvee P \bigwedge x[P \text{ is an inherent property} \wedge x \text{ is a definite thing}) \rightarrow Px].$$

It is sometimes believed that the search for such (non-empty) universal properties residing in (non-empty) things is among the principal tasks of ontology.

Yet another questionable proposition is (c), even if limited to things. Perhaps it is generally true that:

$$\bigwedge x(x \text{ is an object} \equiv x \text{ is non-superdefinite})$$

or, putting it more conveniently:

$$\bigwedge x(x \text{ is non-superdefinite}).$$

This, of course, also applies to empty objects which lack properties and are therefore non-superdefinite (not completely definite).

Let us add that if we accept that:

The Universe = the set of all (non-empty) things,

then the indication of exactly one property that would be universal (for things) would entitle us to assert the weakest *unity* of the universe.

2.

Assume now that:

$$\bigwedge x[x \text{ is a } \textit{supercontradictory} \text{ object } \equiv \bigwedge P(Px \wedge \sim Px)],$$

$$\bigwedge x[x \text{ is a } \textit{contradictory} \text{ object } \equiv \bigvee P(Px \wedge \sim Px)],$$

$$\bigwedge P[P \text{ is a } \textit{supercontradictogenic} \text{ property } \equiv \bigwedge x(Px \wedge \sim Px)]$$

$$\bigwedge P[P \text{ is a } \textit{contradictogenic} \text{ property } \equiv \bigvee x(Px \wedge \sim Px)].$$

Since:

$$\bigwedge x[\bigwedge P(Px \wedge \sim Px) \rightarrow \bigwedge P(Px)]$$

and:

$$\bigwedge x[\bigwedge P(Px \wedge \sim Px) \rightarrow \bigwedge P(Px)],$$

supercontradictory objects are to be found exclusively among superdefinite or indefinite objects. Similarly, in view of:

$$\bigwedge P[\bigwedge x(Px \wedge \sim Px) \rightarrow \bigwedge x(Px)]$$

and:

$$\bigwedge P[\bigwedge x(Px \wedge \sim Px) \rightarrow \bigwedge x \sim (Px)],$$

supercontradictogenic properties are only among universal or non-inherent properties.

We shall use:

$$*$$

as an abbreviation for the expression:

$$Px \wedge \sim Px.$$

The following propositions about supercontradictory and contradictory objects, and supercontradictogenic and contradictogenic properties may be considered:

(a) $\bigwedge x \bigwedge P*$
All objects are supercontradictory
All properties are supercontradictogenic

$\swarrow$ $\searrow$

(b) $\bigvee P \bigwedge x*$
Some properties are supercontradictogenic

(c) $\bigvee x \bigwedge P*$
Some objects are supercontradictory

$\downarrow$ $\downarrow$

(d) $\bigwedge x \bigvee P*$
All objects are contradictory

(e) $\bigwedge P \bigvee x*$
All properties are contradictogenic

$\searrow$ $\swarrow$

(f) $\bigvee x \bigvee X*$
Some objects are contradictory
Some properties are contradictogenic

None of these propositions will survive criticism, despite many attempts to defend them. We should thus accept their negations, especially the strongest one of all – the negation of (f) saying that no object (*sensu stricto*) is contradictory, and no property (*sensu stricto*) is contradictogenic.

Let us call proposition:

$$\bigwedge x \bigwedge P(Px \rightarrow Px)$$

the "identity principle", and proposition:

$$\sim \bigvee x \bigvee P(Px \wedge \sim Px)$$

the "non-contradiction principle", and proposition:

$$\bigwedge x \bigwedge P(Px \vee \sim Px)$$

the "excluded middle principle". If we accept that all three of these principles are equivalent (and bear in mind that not everyone would agree this is so), then the distinction of supercontradictory and non-supercontradictory as well as contradictory and non-contradictory objects on the one hand, and supercontradictogenic and non-supercontra-

dictogenic as well as contradictogenic and non-contradictogenic properties on the other, exhausts the possibilities of classifying objects with regard to whether a given property is resident or not ("$Px \rightarrow Px$", "$Px \wedge \sim Px$", "$Px \vee \sim Px$", where the first and third formulas are equivalent to the negation of the second). As far as this goes, the sequence of propositions (together with negations thereof) reviewed above is exhaustive.

An open question, however, is whether:

$$\bigwedge x \bigwedge P(\sim Px \equiv \text{ non-}Px)$$

and, in general, whether it is admissible to speak of negative properties and their residing in something.

Besides, it is not entirely clear how one should go about describing such properties. If one were to assume that:

$$\bigwedge P[P \text{ is a } \textit{negative} \text{ property } \equiv \bigvee Q \bigwedge x(Px \rightarrow \sim Qx)]$$

then every property would be negative.

3.

Proceeding analogously as in the case of supercontradictory and contradictory objects, we will limit our considerations to the following relations between objects with regard to whether a given property does or does not reside in them:

$$\bigwedge x \bigwedge y[x \text{ is } \textit{like } y \equiv \bigwedge P(Px \rightarrow Py)],$$

$$\bigwedge x \bigwedge y[x \text{ is } \textit{sublike } y \equiv \bigvee P(Px \rightarrow Py)],$$

$$\bigwedge x \bigwedge y[x \text{ is } \textit{supersimilar} \text{ to } y \equiv \bigwedge P(Px \wedge Py)],$$

$$\bigwedge x \bigwedge y[x \text{ is } \textit{similar} \text{ to } y \equiv \bigvee P(Px \wedge Py)].$$

We will also say that two objects are different when they are not alike, taking care not to confuse likeness with the relation of IDENTITY. We thus have:

$$\bigwedge x \bigwedge y[x \text{ is } \textit{different} \text{ from } y \equiv \bigvee P(Px \wedge \sim Py)].$$

Assume at least for non-superdefinite and non-general objects (see below) that:

$$\bigwedge x \bigwedge y \left[\bigwedge P(Px \rightarrow Py) \rightarrow \bigwedge P(Py \rightarrow Px)\right]$$

meaning that if all properties residing in some object also reside in some other object that is not like it, then this other object has no properties besides these. If this is in fact so, we would have:

$$\bigwedge x \bigwedge y[x \text{ is } \textit{sensu stricto} \text{ like } y \equiv \bigwedge P(Px \equiv Py)].$$

Likeness (*sensu stricto*) would thus be a reflexive relation, and moreover every object would be like (*sensu stricto*) itself alone. It is useful to introduce the concept of "likeness *sensu largo*" to denote objects that are not alike in respect to certain selected (and few) properties, such as for example those related to time and space.

Since we have

$$\bigwedge x \bigwedge y \left[\bigwedge P(Px \wedge \sim Px) \rightarrow \bigwedge P(Px)\right]$$

and

$$\bigwedge x \bigwedge y \left[\bigwedge P(Px \wedge \sim Py) \rightarrow \bigwedge P \sim (Py)\right],$$

the only non-subalike objects are objects superdefinite with respect to empty ones.

In turn, because:

$$\bigwedge x \bigwedge y \left[\bigwedge P(Px \wedge Py) \rightarrow \bigwedge P(Px)\right]$$

and:

$$\bigwedge x \bigwedge y \left[\bigwedge P(Px \wedge Py) \rightarrow \bigwedge P(Py)\right],$$

the only supersimilar objects would again be the superdefinite ones.

We also have, *inter alia*:

$$\bigwedge x \bigwedge y(x \text{ is like } y \rightarrow x \text{ is similar to } y).$$

Assume therefore that:

$$\bigwedge x \bigwedge y[x \text{ is similar } \textit{sensu stricto} \text{ to } y$$

$$\equiv (x \text{ is non-like } y \wedge x \text{ is similar to } y)].$$

Let us now collect the propositions about likeness (*sensu stricto*). Of these propositions, (a) together with (b) and (c) equivalent to it are untenable. Theorems (d) and (e), which are also equivalent, should be accepted on the grounds that every object is like itself.

(a) $\bigwedge x \bigwedge y(x \text{ is like } y)$
Everything is like everything
(*versus*: Something is not like something)

/// \\\

(b) $\bigvee x \bigwedge y(x \text{ is like } y)$
Something is like everything
(*vs.*: Everything is non-like something)

$\equiv$

(c) $\bigvee y \bigwedge x(x \text{ is like } y)$
There is something to which everything is alike
(*vs.*: For everything there is something to which it is non-alike)

↓ ↓

(d) $\bigwedge y \bigvee x(x \text{ is like } y)$
For everything there is something to which it is non-like
(*vs.*: There is something nothing to which nothing is alike)

$\equiv$

(e) $\bigwedge x \bigvee y(x \text{ is like } y)$
Everything is like something
(*vs.*: Something is not like anything)

↘ ↙

(f) $\bigvee x \bigvee y(x \text{ is like } y)$
Something is like something
vs.: Nothing is like anything)

For similarity we have:

(a) $\bigwedge x \bigwedge y(x \text{ is similar to } y)$
Everything is similar to everything
(*vs.*: Something is dissimilar from something)

↙ ↘

(b) $\bigvee x \bigwedge y(x \text{ is similar to } y)$
Something is similar to everything
(*vs.*: Everything is dissimilar to something)

$\equiv$

(c) $\bigvee y \bigwedge x(x \text{ is similar to } y)$
There is something to which everything is similar
(*vs.*: For everything there is something to which it is dissimilar)

↓ ↓

(d) $\bigwedge y \bigvee x(x \text{ is similar to } y)$
For everything there is something to which it is similar
(*vs.*: There is something to which nothing is similar)

$\equiv$

(e) $\bigwedge x \bigvee y(x \text{ is similar to } y)$
Everything is similar to something
(*vs.*: Something is not similar to anything)

$\searrow$ $\swarrow$

(f) $\bigvee x \bigvee y(x \text{ is similar to } y)$
Something is similar to something
(*vs.*: Nothing is similar to anything)

Here neither (b) or (c) imply proposition (a) since in the former two cases similarity may be in respect to different properties (the similarity relation is intransitive). If some property were universal, then of course the strongest proposition (a) would have to be accepted. If one were to consider only similarity *sensu stricto*, then only the weakest proposition (f) might perhaps be acceptable. Propositions (d) and (e) would be satisfied without this limitation since it would be permissible to recognize every object as similar to itself.

The rejection of all the propositions (a)–(f), and hence the acceptance of the negation of (f), would be a consequence of the view that each object is exceptional and each property singular (see below).

4.

Let us introduce the following definitions:

$$\bigwedge x \, \{x \text{ is an } \textit{exceptional} \text{ object}$$
$$\equiv \bigwedge P \bigwedge y \, [(x \text{ is non-identical with } y \wedge Px) \rightarrow \sim Py]\Big\},$$

$$\bigwedge x \, \{x \text{ is a } \textit{subexceptional} \text{ object}$$
$$\equiv \bigwedge P \bigvee y[(x \text{ is non-identical with } y \wedge Px) \rightarrow \sim Py]\Big\},$$

$$\bigwedge x \, \{x \text{ is an } \textit{individual} \text{ object}$$
$$\equiv \bigvee P \bigwedge y[(x \text{ is non-identical with } y \wedge Px) \rightarrow \sim Py]\Big\},$$

$$\bigwedge x \, \{x \text{ is a } \textit{subindividual} \text{ object}$$

$$\equiv \bigvee P \bigvee y[(x \text{ is non-identical with } y \wedge Px) \rightarrow {\sim} Py]\Big\},$$

$$\bigwedge P \{P \text{ is a } \textit{singular} \text{ property}$$
$$\equiv \bigwedge x \bigwedge y[(x \text{ is non-identical with } y \wedge Px) \rightarrow {\sim} Py]\Big\},$$

$$\bigwedge P \{P \text{ is a } \textit{subsingular} \text{ property}$$
$$\equiv \bigvee x \bigwedge y[(x \text{ is non-identical with } y \wedge Px) \rightarrow {\sim} Py]\Big\},$$

$$\bigwedge P \{P \text{ is a } \textit{limited} \text{ property}$$
$$\equiv \bigwedge x \bigvee y[(x \text{ is non-identical with } y \wedge Px) \rightarrow {\sim} Py]\Big\},$$

$$\bigwedge P \{P \text{ is a } \textit{sublimited} \text{ property}$$
$$\equiv \bigvee x \bigvee y[(x \text{ is non-identical with } y \wedge Px) \rightarrow {\sim} Py]\Big\}.$$

As we can see, an exceptional object is such that none of its properties resides in any other object, meaning that each of its properties is singular. On the other hand, an individual object is such that each of its properties does not reside in a certain other object, being a limited property. An object is individual when it has some property that does not reside in any other object, that is when it has a singular property.[4] A subindividual object is distinguished by some (limited) property that does not reside in a certain other object. A singular property is one which resides in one object at most (it resides exclusively in the object which has it). A limited property is such that does not reside in at least one object, which makes this property non-universal.

Singular and subsingular properties are indistinguishable, as are limited and sublimited ones. There can be no more than one object having a certain property exclusively. Similarly, a property of an object is limited when it remains as it is no matter which object it resides in (as long as this property is non-universal).

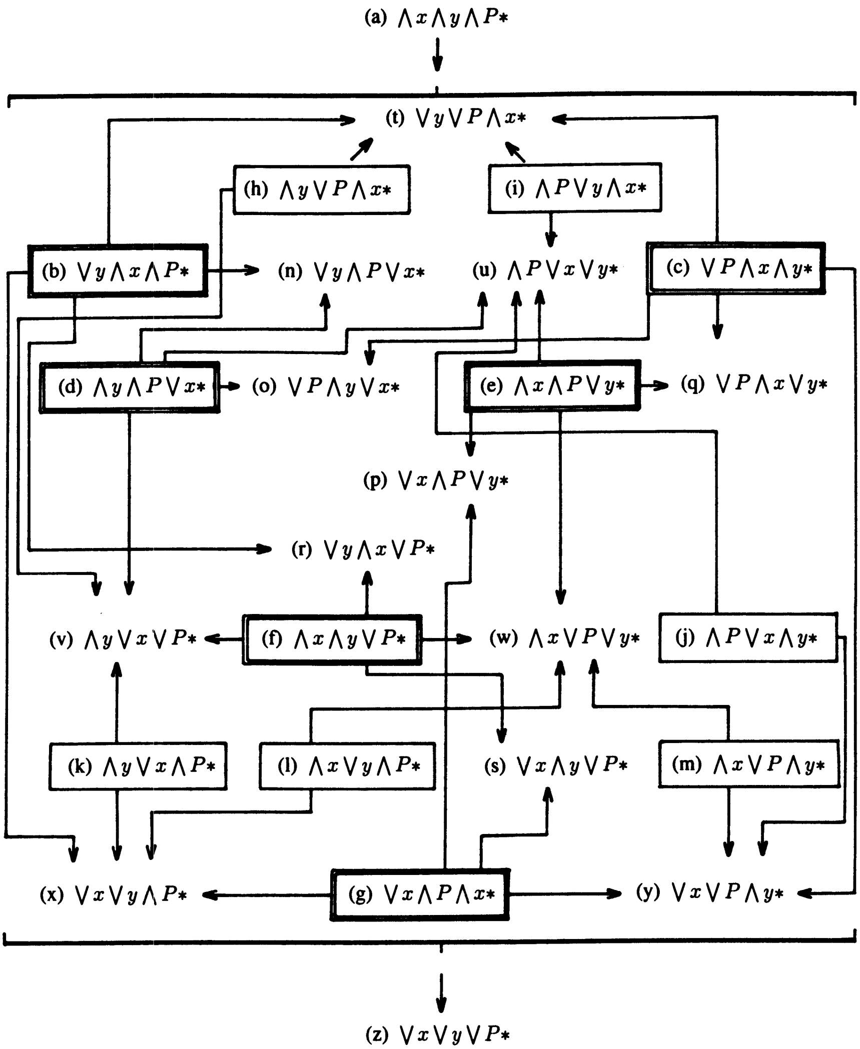
Let

$$*$$

stand for

$$(x \text{ is unidentical with } y \wedge Px) \rightarrow Py.$$

We have the following propositions:


(a) $\bigwedge x \bigwedge y \bigwedge P*$
(t) $\bigvee y \bigvee P \bigwedge x*$
(h) $\bigwedge y \bigvee P \bigwedge x*$
(i) $\bigwedge P \bigvee y \bigwedge x*$
(b) $\bigvee y \bigwedge x \bigwedge P*$
(n) $\bigvee y \bigwedge P \bigvee x*$
(u) $\bigwedge P \bigvee x \bigvee y*$
(c) $\bigvee P \bigwedge x \bigwedge y*$
(d) $\bigwedge y \bigwedge P \bigvee x*$
(o) $\bigvee P \bigwedge y \bigvee x*$
(e) $\bigwedge x \bigwedge P \bigvee y*$
(q) $\bigvee P \bigwedge x \bigvee y*$
(p) $\bigvee x \bigwedge P \bigvee y*$
(r) $\bigvee y \bigwedge x \bigvee P*$
(v) $\bigwedge y \bigvee x \bigvee P*$
(f) $\bigwedge x \bigwedge y \bigvee P*$
(w) $\bigwedge x \bigvee P \bigvee y*$
(j) $\bigwedge P \bigvee x \bigwedge y*$
(k) $\bigwedge y \bigvee x \bigwedge P*$
(l) $\bigwedge x \bigvee y \bigwedge P*$
(s) $\bigvee x \bigwedge y \bigvee P*$
(m) $\bigwedge x \bigvee P \bigwedge y*$
(x) $\bigvee x \bigvee y \bigwedge P*$
(g) $\bigvee x \bigwedge P \bigwedge x*$
(y) $\bigvee x \bigvee P \bigwedge y*$
(z) $\bigvee x \bigvee y \bigvee P*$

These propositions illustrate issues which continue to be objects of controversy. The principal positions in this controversy are, on the one hand, that of accepting proposition (a), and on the other hand, that of rejecting propositions (c) (Some properties are singular), (g) (Some objects are exceptional), (q) (Some properties are limited), and (z) and hence also (a). The controversy is part of one form of the controversy about universals[5] concerning the question of what corresponds to predicables in sentences like "This object here is red": the redness-of-this-object-here, i.e. a certain singular property, or redness-in-general, i.e. a general (common) property? In the former case, residing would consist in a kind of coalescence of the entirety of the given property with the other properties residing in this same object. In the latter case, according to one approach, redness would be resident in a given object if a *part* of this redness were fused with relevant parts of the remaining properties residing in this object. According to another view, the question of properties and general objects in general would have this form:

$$\bigwedge x \bigwedge y \bigwedge z \langle x \text{ is a generalization of } y \text{ and } z$$
$$\equiv \big\{(x \text{ is unidentical with } y \wedge x \text{ is unidentical with } z \wedge y$$
$$\text{is unidentical with } z) \wedge \bigwedge P[(Py \wedge Pz) \equiv Px]\big\}\rangle.$$

Analogously,

$$\bigwedge x[x \text{ is a } \textit{general} \text{ object}$$
$$\equiv \bigvee y \bigvee z(x \text{ is a generalization of } y \text{ and } z)].$$

Note: General objects may be generalizations of more than two objects falling under them.

Such a treatment of general objects is vulnerable to various criticisms.

Firstly, objects falling under every general object would be general *ex definitione*. Secondly, if any one of them had, for example, some singular property, then, *ex definitione*, it would also have a corresponding negated property since the general object would have it. If the implication:

$$\ldots \bigwedge P[(Py \wedge Pz) \rightarrow Px]$$

(standing for the last part of the formula concerning generalization) were assumed to rule out these consequences, then a generalization of two

arbitrary objects similar in some respect would be every object similar to them in this particular respect. If we put here:

$$\ldots \bigwedge P\,\{Px \equiv [(Py \wedge Pz) \vee \bigwedge x'(x' \text{ is a general object } \rightarrow Px')]\},$$

then we would have an (indirect) vicious circle.

It would thus seem that general objects are at most *quasi*-objects.[6]

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NOTES

[1] I agree with Leon Chwistek's opinion that ontological language requires not explanation or rearrangement but total reconstruction. This postulate is the essense of Chwistek's constructive method, being congenial to Kazimierz Twardowski's method of analytical description. *Cf.* my papers: "On Leon Chwistek's semiotic views", in J. Pelc *et al.* (Eds.), *Sign, System and Function*, Mouton Publishers, Berlin, 1984, pp. 77–87; "Kazimierz Twardowski's descriptive semiotics", *Poznań Studies in the Philosophy of the Sciences and the Humanities* (in press). Such a reconstruction of ontological language has a long tradition in Poland. *Cf.* my paper "On the sources of contemporary Polish logic", *Dialectics and Humanism* **4**, 1990, 163–183. *Cf.* also my paper "Definition, explication, and paraphrase in Ajdukiewiczian tradition", in *Ajdukiewicz on Language and Meaning. Proceedings of the International Symposium in the Centenary of Kazimierz Ajdukiewicz's Birth* (in press).

[2] I deny existence (but not being!) of mental objects. *Cf.* my papers: "Being and existence. On the being of what seems not to be", *Dialectics and Humanism* **4**, 1981, 131–139; "De la soi-disant théorie des descriptions", *Kwartalnik Neofilologiczny* **1**, 1987, 81–91.

[3] The ontological structure of mental (*sc.* purely intentional) objects is very precisely arranged by Roman Ingarden. *Cf.* my paper "On Roman Ingarden's semiotic views", *Analecta Husserliana*, **27**, 1989, 523–540.

[4] The term "individual object" is understood in this way *i.a.* by Kazimierz Twardowski. *Cf.* my work "The metaphysical basis of Kazimierz Twardowski's descriptive semiotics", in F. Coniglione, R. Poli and J. Woleński (Eds.), *Polish Scientific Philosophy*, Editions Rodopi, Atlanta, 1993, pp. 191–206.

[5] I analyze different versions of this controversy in my paper "Controversy about universals", in T. A. Sebeok (Gen. Ed.), *Encyclopedic Dictionary of Semiotics*, Mouton; De Gruyter, Berlin, 1986, pp. 1138–1141.

[6] This paper is the introductory chapter of a larger work *Ontological Mininum*, prepared for Editions Rodopi, Atlanta.

GRZEGORZ MALINOWSKI

INFERENTIAL MANY-VALUEDNESS

The problem of interpretation of logical values in addition to truth and falsity is still among the most controversial questions of contemporary logic. In connection with this the investigations of logical formalizations bore several descriptions of many-valued constructions in terms of zero-one valuations, cf. e.g. [6], [10] and [11]. Effectively, the interpretations associated with these descriptions shed new light on the problem of logical many-valuedness.

The present article contributes to the question of the nature of many-valuedness. Its main aim is to complete some ideas traced out by R. Suszko in [7]–[10]. The most essential feature of the approach is the opposition of inferential two-valuedness to referential many-valuedness. Accepting Suszko's thesis which states that each logic, i.e. a structural consequence operation conforming to Tarski's conditions, is logically two-valued we show a way to get inferential many-valuedness.

1. PROPOSITIONAL LANGUAGE AND ITS INTERPRETATION STRUCTURES

Let *Var* = $\{p, q, r, \ldots\}$ be a denumerable set of propositional variables and $F = \{F_1, \ldots, F_m\}$ a finite set of sentential connectives – with each connective F_i there is a natural number $a(F_i)$ associated that describes its syntactical category (i.e. number of arguments). Assume, moreover, that $a(F_i) \neq 0$ for some $i \in \{1, \ldots, m\}$. Define inductively the set of formulas, *For*, putting

(i) $Var \subseteq For$

(ii) For any $F_i \in F$ such that $a(F_i) = k$, $F_i(\alpha_1, \ldots, \alpha_k) \in For$ whenever $\alpha_1, \ldots, \alpha_k \in For$.

The algebra of formulas

$$\mathcal{L} = (For, F_1, \ldots, F_m)$$

Jan Woleński (ed.), Philosophical Logic in Poland, 75–84.

constructed in such a way will be called a *propositional language*.

When interpreting a language each formula Φ is provided with a meaning which is its *semantic correlate*. Two conditions are required as far as a mapping r of *For* into the range A of all semantic correlates is concerned:

(*) With each $\Phi \in For$ exactly one semantic correlate is associated, i.e. r is a function;

(**) Two formulas $\alpha, \beta \in For$ are interchangeable in any propositional context $\Phi \in For$ whenever $r(\alpha) = r(\beta)$ i.e. for any $\Phi \in For, p \in Var$

$$r(\Phi(\alpha/p)) = r(\Phi(\beta/p)) \quad \text{if and only if} \quad r(\alpha) = r(\beta),$$

where $\Phi(\alpha/p)$ and $\Phi(\beta/p)$ stand for the formulas resulting from Φ after the substitution of $\alpha(\beta)$ for p.

Conditions (*), (**) were assumed by Frege [2]. Since Frege identified the semantic correlates of propositions with the logical values of truth and falsity he inevitably tended towards classical logic. Note that (**) is an exemplification of Leibniz *principle of extensionality* and states that the denotation of a proposition is
a function of the denotations of its components.

1.1. (cf. [8]). When A is the set of all semantic correlates of the language $\mathcal{L} = (For, F_1, \ldots, F_m)$, then for each $i = 1, \ldots, m$ the formula

$$f_i(r(\alpha_1), \ldots, r(\alpha_n)) = r(F_i(\alpha_1, \ldots, \alpha_n))$$

defines uniquely a function f_i on A of the same arity as F_i.

From 1.1 it immediately follows that each interpretation structure

$$\mathcal{A} = (A, f_1, \ldots, f_n)$$

is an algebra similar to $\mathcal{L}$. In turn, any mapping $r : Var \rightarrow A$ may be extended uniquely to the homomorphism $h_r : \mathcal{L} \rightarrow \mathcal{A} (h_r \in Hom(\mathcal{L}, \mathcal{A}))$ and, therefore, $\mathcal{L}$ is an *absolutely free* algebra in its similarity class – it is generated by the set *Var*. In addition, interpretations of propositional languages are determined by algebraic substitutions of propositional variables. In their turn functions of the logical algebras $\mathcal{A}$ may be described in an arbitrary way accepted in algebra, including tables.

1.2.E. The structure $\mathcal{B}_2 = (\{0, 1\}, \neg, \rightarrow, \vee, \wedge, \equiv)$ with the operations defined by truth-tables is the (special) algebra of the classical logic.

Propositional languages may also serve as special interpretation structures. $\mathcal{L}$ as well as any language similar to it are such structures. In the former case the endomorphisms $e : \mathcal{L} \to \mathcal{L}$ referred to in the sequel as *substitutions* take the role of interpretation mappings.

2. MATRICES AND STRUCTURALITY

Interpretation structures equipped with a distinguished subset of the set of semantic correlates corresponding to propositions of a specified kind (e.g. *true propositions*) are called logical *matrices*. More specifically, a pair

$$M = (\mathcal{A}, D),$$

with $\mathcal{A}$ being an algebra similar to a propositional language $\mathcal{L}$, and $D \subseteq A$ a nonempty subset of the universe of $\mathcal{A}$ will be referred to as *matrix* for $\mathcal{L}$. Elements of D will be called *designated elements* of M. With each matrix M for $\mathcal{L}$ there is associated a set of formulas which take designated values only:

$$E(M) = \{\alpha \in For : h\alpha \in D \quad \text{for any} \quad h \in Hom(\mathcal{L}, \mathcal{A})\}$$

called its *content*.

2.1.E. $M_2 = (\mathcal{B}_2, \{1\})$ is the matrix for CPC (i.e. classical propositional calculus). Clearly, its content coincides with the set of all tautologies, $TAUT$, $E(M_2) = TAUT$.

The notion of *matrix consequence* is a natural generalization of the classical consequence and is defined as follows: the relation $\models_M \subseteq 2^{For} \times For$ is said to be a *matrix consequence* of M provided that for any $X \subseteq For$, $\alpha \in For$

$$X \models_M \alpha \quad \text{if and only if} \quad \text{for every} \quad h \in Hom(\mathcal{L}, \mathcal{A}) \\ (h\alpha \in D \quad \text{whenever} \quad hX \subseteq D).$$

Notice that

$$E(M) = \{\alpha : \emptyset \models_M \alpha\}.$$

In turn, with every $\models_M$ there may be uniquely associated an operation $Cn_M : 2^{For} \to 2^{For}$ such that

$$\alpha \in Cn_M(X) \quad \text{if and only if} \quad X \models_M \alpha.$$

Where K is a class (a set) of matrices for a given language $\mathcal{L}$, the relation $\models_K$ is to be identified with the set-theoretical meet of $\{\models_M : M \in K\}$. Consequently, $Cn_K = \bigcap\{Cn_M : M \in K\}$ i.e. for any

$X \subseteq For$

$$Cn_K(X) = \bigcap\{Cn_M(X) : M \in K\}.$$

Cn_M and Cn_K are special examples of the operations considered in the general theory of deductive systems originating with Tarski [1]. A mapping $C : 2^{For} \to 2^{For}$ will be referred to as a *consequence operation* of the language $\mathcal{L}$ if and only if for any $X, Y \subseteq For$

(T0) $X \subseteq C(X)$

(T1) $C(C(X)) = C(X)$

(T2) $C(X) \subseteq C(Y)$ whenever $X \subseteq Y$.

If, moreover, for any substitution $e \in End(\mathcal{L})$

S $eC(X) \subseteq C(eX)$,

we shall say that C is *structural.*

It is easy to prove that each matrix consequence operation Cn_M is structural. Conversely, each structural consequence C of $\mathcal{L}$ and any set of formulas X determine together a matrix

$$L = (\mathcal{L},\ C(X))$$

called a *Lindenbaum matrix.* The class of all Lindenbaum matrices of a given consequence C of $\mathcal{L}$,

$$LC = \{(\mathcal{L}, C(X)) : X \subseteq For\}$$

will be referred to as *Lindenbaum bundle.* Since the substitutions (i.e. endomorphisms) of the language $\mathcal{L}$ take then the role of valuations one may easily show that any structural C is uniquely determined by its Lindenbaum bundle, $C = Cn_{LC}$ and ultimately

2.2. (cf. [12]). For every structural consequence operation C there exists a class K of matrices such that $C = Cn_K$.

3. SUSZKO'S THESIS

Suszko in [9] calls attention to the referential character of homomorphisms associating with propositions their (possible) semantic correlates. Subsequently, he opposes them to *logical valuations*, which are

zero-one valued functions defined on *For*. Given a propositional language $\mathcal{L}$ and a matrix $M = (\mathcal{A}, D)$ for $\mathcal{L}$, the set of valuations TV_M is defined as:

$$TV_M = \{t_h : h \in Hom)\},$$

where

$$t_h(\alpha) = \begin{cases} 1 & \text{if} \quad h(\alpha) \in D \\ 0 & \text{if} \quad h(\alpha) \notin D \end{cases}.$$

Notice that $card(TV_M) \leq card(Hom(\mathcal{L}, \mathcal{A})$ (in general, $h_1 \neq h_2$ does not imply that $t_{h_1} \neq t_{h_2}$!). Notice, moreover, that

[T] $X \models_M \alpha$ if and only if for every $t \in TV_M$ $t(\alpha) = 1$ whenever $t(X) \subseteq \{1\}$.

The definition of logical valuations can be simply repeated with respect to any structural consequence operation C (or, equivalently, for any relation $\vdash_C$ associated with C) since according to 2.2 for each such C there is a class of matrices K having the property that

$$C = \bigcap\{Cn_M : M \in K\},$$

Thus, *each structural logic* $(\mathcal{L}, C)$ can be determined by a class of logical valuations of the language $\mathcal{L}$ or, in other words, *it is logically two-valued*, cf. [9].

The justification of *Suszko's thesis* that states logical two-valuedness of an important family of logics lacks the description of valuations (i.e. elements of TV_C) for an arbitrary relation $\vdash_C$. An example of a relatively easily definable set of logical valuations is $LV3$, the class adequate for the $(\rightarrow, \neg)$-version of the three-valued Łukasiewicz logic (cf. [3]). $LV3$ is the set of all functions $t : For \rightarrow \{0, 1\}$ such that for any $\alpha, \beta, \gamma \in For$ the following conditions hold:

(0) $t(\gamma) = 0$ or $t(\neg\gamma) = 0$

(1) $t(\alpha \rightarrow \beta) = 1$ whenever $t(\beta) = 1$

(2) if $t(\alpha) = 1$ and $t(\beta) = 0$, then $(\alpha \rightarrow \beta) = 0$

(3) if $t(\alpha) = t(\beta)$ and $t(\neg\alpha) = t(\neg\beta)$, then $t(\alpha \rightarrow \beta) = 1$

(4) if $t(\alpha) = t(\beta) = 0$ and $t(\neg\alpha) \neq t(\neg\beta)$, then $t(\alpha \to \beta) = t(\neg\alpha)$

(5) if $t(\neg\alpha) = 0$, then $t(\neg\neg\alpha) = t(\alpha)$

(6) if $t(\alpha) = 1$ and $t(\beta) = 0$, then $t(\neg(\alpha \to \beta)) = t(\neg\beta)$

(7) if $t(\alpha) = t(\neg\alpha) = t(\beta)$ and $t(\neg\beta) = 1$, then $t(\neg(\alpha \to \beta)) = 0$.

cf. [10].

Usually, the degree of complexity of a many-valued logic's description increases with the number of values. But in some cases, it can be simplified by the application of extra connectives "identifying" original (matrix) values. Such a use of j-operators of Rosser and Turquette makes it possible to obtain e.g. a uniform description of valuations for finite Łukasiewicz logics, cf. [4]. It should be said, however, that the problem of finding a general method permitting one to describe sets of valuations for arbitrary logic remains a difficult open question.

Suszko's valuation procedure forms a part of a broader semantical programme related to the conception of so-called non-Fregean logics (cf. [7]). According to that programme there are situations which play the role of semantic correlates of propositions. Logical valuations, for their part, are characteristic functions of the sets of formulas which are counterimages of the sets of positive situations, i.e. of those which obtain, under homomorphisms establishing the interpretation. Following Suszko one may say e.g. that n-valued Łukasiewicz or Post logic (n finite) is a two-valued logic of n situations. Obviously, then situations which are referential concepts must not be confused with the logical values, in particular with falsity and truth.

4. LOGICAL THREE-VALUEDNESS

The logical two-valuedness discussed in Section 3 is obviously related to the division of the universe of interpretation into two subsets of elements: distinguished and others. It also turned out, under the assumption of structurality, that Tarski's concept of consequence may be considered as a "bivalent" inference operation. One may then naturally ask whether logical many-valuedness is possible at all. Below, we give

an affirmative answer to this question by invoking a formal framework for reasoning admitting rules of inference which lead from non-rejected assumptions to accepted conclusions.

The central notions of the framework, exhaustively discussed in [5], are counterparts of the concepts of matrix and consequence – both distinguished by the prefix "q" which may be read as "quasi". Where $\mathcal{M} = (M, F_1, F_2, \ldots, F_n)$ is an algebra similar to a given propositional language $\mathcal{L}$ and $\overline{D}, D$ are disjoint subsets of $M(\overline{D} \cap D = \emptyset)$ the triple

$$M = (\mathcal{M}, \overline{D}, D)$$

will be called a *q-matrix* for $\mathcal{L}$. $\overline{D}$ and D may be then interpreted as sets of *rejected* and *distinguished* values of M, respectively.

For any such M we define the *q-consequence relation* $\models^*{}_M \subseteq 2^{For} \times For$ as follows:

$$X \models^*{}_M \alpha \quad \text{if and only if} \quad \text{for every} \quad h \in Hom(\mathcal{L}, \mathcal{M}) \quad (hX \cap \overline{D} = \emptyset \quad \text{implies} \quad h\alpha \in D).$$

Obviously, with each relation $\models^*{}_M$ one may associate the operation $Wn_M : 2^{For} \to 2^{For}$ putting

$$Wn_M(X) = \{\alpha : X \models^*{}_M \alpha\}.$$

Notice that when $\overline{D} \cup D = M, Wn_M$ coincides with the consequence Cn_M determined by the matrix $M = (\mathcal{M}, D)$. In other cases the two operations differ from each other – to see this consider any q-matrix of the form $(\{e_1, e_2, e_3\}, f_1, f_2, \ldots, f_n, \{e_1\}, \{e_3\})$.

It is easy to observe that for any q-matrix M for which $\overline{D} \cup D \neq M$ no class TV of functions t : $For \to \{0, 1\}$ exists such that for all X and $\alpha, X \models^*{}_M \alpha$ iff for each $t \in TV$ $(t(X) \subseteq 1$ implies $t\alpha = 1)$. Thus, some Wn_M are not logically two-valued (cf. Section 3).

Now, for every $h \in Hom(\mathcal{L}, \mathcal{M})$ let us define a three-valued function k_h : $For \to \{0, 1/2, 1\}$ putting

$$k_h(\alpha) = \begin{cases} 0 & \text{if} \quad h(\alpha) \in \overline{D} \\ 1/2 & \text{if} \quad h(\alpha) \in M - (\overline{D} \cup D) \\ 1 & \text{if} \quad h(\alpha) \in D \end{cases}.$$

Given a q-matrix M for $\mathcal{L}$ let $KV_M = \{k_h : h \in Hom(\mathcal{L}, \mathcal{M})\}$; we obtain

[K] $X \models^*{}_M \alpha$ if and only if for every $k_h \in KV_M$
$(k_h(X) \cap \{0\} = \emptyset$ implies
$k_h(\alpha) = 1)$.

This is a kind of three-valued description of the q-consequence relation $\models^*{}_M$. Notice that KV_M reduces to TV_M as well as [K] to [T] when $\overline{D} \cup D = M$.

In [5] the concept of q-consequence operation of which Wn_M is a prototype was introduced and studied. An operation $W : 2^{For} \to 2^{For}$ is a *q-consequence* provided that for every $X, Y \subseteq For$

(W1) $W(X) \subseteq W(Y)$ whenever $X \subseteq Y$

(W2) $W(X \cup W(X)) = W(X)$.

W is called *structural* if for every substitution $e \in End(\mathcal{L})$

(S) $eW(X) \subseteq W(eX)$.

Where M is any q-matrix, Wn_M is structural. In turn, all Lindenbaum's tools may be adopted to structural q-consequence operations W to exactly the same effect. Thus, the bundle of Lindenbaum's q-matrices which are of the form

$$W_X = (For, For - (X \cup W(X)), W(X)),$$

compare 2.2, may be used to prove

4.1. (see [5]). For every structural q-consequence W there exists a class of q-matrices K such that $W = \bigcap\{Wn_M : M \in K\}$.

Therefore, we may conclude that *each (structural) logic* $(\mathcal{L}, W)$ *is logically two or three-valued.* Clearly, logically three-valued logics exist : consider any q-logic $(\mathcal{L}, Wn_M)$ for which $Wn_M \neq Cn_M$.

5. TOWARDS THE CONCEPT OF LOGICAL MANY-VALUEDNESS

Inferential three-valuedness discussed above was grounded on a generalization of Tarski's concept of consequence operation. It is also in order to remark that this *inferential* conception is entirely consistent with the common understanding of logical system as a set of formulas closed

under substitutions, usually defined as a content of a logical matrix:

5.1. For any q-matrix $M = (\mathcal{M}, \overline{D}, D)$ and a corresponding matrix $M^* = (\mathcal{M}, D), Cn_{M^*}(\emptyset) = Wn_M(\emptyset) = E(M^*)$.

This means that any logical system may equally well be extended to two-valued logic $(\mathcal{L}, Cn_{M^*})$ or to a three-valued logic $(\mathcal{L}, Wn_M)$. Then, obviously, depending on the quality and cardinality of M both kinds of extensions may take different shapes, thus defining different logics. Perhaps the most striking is that even CPC, i.e. the content of two-element (or two-valued) matrix M_2 can be extended to the three-valued q-inference. To complete the view we should add that this extension is trivial:

5.2.E. The q-matrix $qM_2 = (\mathcal{B}_2, \emptyset, \{1\})$ determines the q-consequence operation Wn_{qM_2} such that $Wn_{qM_2}(\emptyset) = E(M_2)$. However, the "inferential" part of this q-consequence is uninteresting since the class of non-axiomatic rules comprises all sequents of the form $X \vdash \alpha$, where α is a tautology.

Obviously, for some inferentially three-valued logics based on three-element algebras, referential assignments i.e. algebraic homomorphisms (cf. Section 1) $h \in Hom(\mathcal{L}, \mathcal{A})$ and logical valuations $k_h \in V_M$ do coincide. This phenomenon delineates a special class of logics, which satisfy a "generalized" version of the Fregean Axiom identifying in one-one way three logical values and three semantical correlates (or referents).

The logical three-valuedness set forth in Section 4 is clearly mirrored in the construction of the q-matrices: the universe of any q-matrix is divided into three subsets. And, similarly as in the two-valued case, logical three-valuedness relates to that division, which obviously tempts us to introduce the notions of logical n-valuedness (for each natural $n > 3$). On the whole the solution being intuitive and to some extent natural looks quite promising. Why not to discern e.g. different degrees of rejection and further why not to define "matrix" inference relation and operation in a very much similar manner? A little more that I can say about the last step is left till another occasion. The reader, however, is invited to consult [5] and to pose the question as to how the *many-valued* inference he has in mind might be characterized deductively i.e. through the set of rules of inference and appropriate conception of proof. One will then quickly point out that the notion of proof introduced for q-consequence operation is the weakest possible when we retain the *usual* notion of a rule of inference. To give an idea we

may say that it differs essentially from the standard proof in exactly one point: the *repetition rule* $\{\alpha/\alpha : \alpha \in For$ is no longer unrestrictedly accepted as one of the consequences of the *postulate* that each premise (or assumption) in the proof is automatically accepted as a conclusion (a subsequent step in the proof). Perhaps then, a slightly better way of providing a workable framework for obtaining logical many-valuedness is to change the concept of a rule of inference.

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A JAŚKOWSKI-STYLE SYSTEM OF COMPUTER-ASSISTED REASONING

PART ONE: ON JAŚKOWSKI AND EUCLID

1. Historical Preliminaries

Jan Łukasiewicz and Stanislaw Jaśkowski are credited with being the first who remarked that practising mathematicians do not appeal to logic theories but make use of other methods of reasoning. They meant those logical theories which were worked out by Russell, Frege, Peano, Hilbert, and Polish logicians, including Łukasiewicz himself. According to Jaśkowski's testimony, in 1926 Łukasiewicz raised the problem of how to build a logical system that would agree with the practice of employing suppositions in proving theorems. Jaśkowski (1934) offered a solution of this problem in his booklet *On the Rules of Supposition in Formal Logic.*

The actual term *natural deduction* does not appear in Jaśkowski's paper; instead, he uses the phrase *the method of supposition.* It was rather Gentzen (1934–1935) who introduced this term in the German version *das natürliche Schließen.* To complete the story of the origins, let it be recalled that the first communication on suppositional proofs appeared in the proceedings of the first Polish Mathematical Congress (held in 1927) entitled *Księga pamiątkowa pierwszego polskiego zjazdu matematycznego*, Kraków 1929; in this note Jaśkowski reported on results presented at Łukasiewicz's seminar in 1926.

Of the two authors who in the same year 1934, independently of each other, initiated the method of natural deduction, Gerhard Gentzen proved to be the more successful. His work has won much wider renown. This may be partly due to the fact that Gentzen's system of rules was more conspicuous and elegant with its two symmetric sets of rules, those for the introduction of logical constants, and those for the elimination of constants. Another explanation can be found in the impact of context; namely, in the same paper the presentation of natural

Jan Woleński (ed.), Philosophical Logic in Poland, 85–101.

deduction was accompanied by the introduction and discussion of the famous *Hauptsatz* which contributed so much to the development of 20th century logic, also in its computerized form.

Jaśkowski's tradition was continued in Poland, especially in the textbook by Słupecki and Borkowski (1967), and other textbooks by these authors; let both the book in question and the system it contains be referred to as SB. There is a remark in SB that it belongs to natural deduction systems as invented by Jaśkowski and by Gentzen, and that it differs from each of them in some detail. In fact, though SB resembles Gentzen in the division of inference rules into the introduction set and the elimination set, and it resembles Jaśkowski in a method of structuring proofs, it has some original features which are something more than insignificant details. However, this is a special story that should be told elsewhere.

There is a feature in Jaśkowski's system which is alien to SB, and which so far remains fairly unnoticed, in spite of its being closely related to the idea of rendering mathematical practice with a logical theory. Let us discuss it briefly in the next section.

2. *Whether Natural Deduction May Become More Natural*

Though the method of natural deduction as found in Gentzen, in SB, etc., brings us closer to actual methods of reasoning, there is something in it that can hardly be called natural. It is usual in such systems that at the right margin of each proof line one makes reference to both the premises from which this line derives and the rule which justifies the derivation. This usage is very convenient, indeed, and may be specially recommended for didactic purposes in teaching logic. However, if a formalization is to reconstruct natural inferences as truly as possible, the method of referring to rules should somehow be altered. This requires a new kind of formalization in which making use of inference rules would more resemble those linguistic means which occur in the mathematical vernacular.

The need for such a formalization might not have been felt until computers proved ready for either simulating or checking our intellectual activities, including the proving of theorems. Once such possibilities have opened, one may expect a device for writing mathematical texts in the familiar traditional form, as best suiting our habits, the same form being logically verifiable with a computerized algorithm (i.e., a

suitable software). Obviously, such a device could have been invented without any influence of Jaśkowski's ideas, but a comparison with those ideas provides us with useful heuristic hints. So, before I report on a system belonging both to natural deduction and to *Computer-Assisted Reasoning*, CAR for short, it seems worth while to recall Jaśkowski's precursory approach.

Both in the CAR system to be discussed and in Jaśkowski's work there are no free variables (called also real variables) in predicate logic, by contrast with, e.g. SB. In both systems this feature is related to the method of referencing rules. This method does not consist of putting rule names outside proof lines as metalinguistic comments; instead one inserts suitable expressions inside the very text of the proof in question, as is usual in non-formalized, and in this sense natural, mathematical texts. Here is the relevant comment of Jaśkowski as far as the concept of variable is concerned.

Symbols of variables which are not apparent variables do not merit the name of variable at all. We deal with such a term as with a given constant, though it is neither a primitive term nor a defined one. It is a constant, the meaning of which, although undefined, remains unaltered through the whole process of reasoning. In practice, we often introduce such undefined constants in the course of a proof. For example, we say: "Consider an arbitrary "x" and then we deduce propositions which can be said to belong to the *scope of constancy* of the symbol "x". This process of reasoning will be applied in our system. (p. 28)

The procedure described above consists of the operation of eliminating a quantifier (what quantifier will be seen in the example discussed below), and this amounts to replacing a bound variable with a constant. However, instead of referencing a rule in the metalinguistic style, one inserts an expression in the language of the proof itself, viz. the word '*consider*'. For proofs formalized in this system, Jaśkowski suggests imitating this word with the symbolic constant 'T' which is analogous to the constant 'S', for '*Suppose*', introduced previously (at the level of propositional logic). We may guess that the letter 'T' was chosen by the author to hint at the fact of substituting a *term* for a variable. The scope of constancy of the term 'T' is called its *domain*; in this domain the meaning of the term is to be regarded as fixed. The author exemplifies his point with a proof of the proposition.

$$C\Pi x\Pi y\phi xy\Pi z\phi zz.$$

Here is the proof:

(1) $1 \cdot S\Pi x\Pi y\phi xy$

Now we write "Tz" for "Consider an arbitrary z":

(2) $1 \cdot 1 \cdot Tz$
(3) $1 \cdot 1 \cdot \Pi y \phi zy$
(4) $1 \cdot 1 \cdot \phi zz$
(5) $1 \cdot \Pi z \phi zz$
(6) $C\Pi x \Pi y \phi y \Pi z \phi zz$

It is seen in this example that the constant "T" implicitly references the rule of eliminating the universal quantifier. The same can be done through explicit reference in the usual metalinguistic style, but Jaśkowski's method closer approximates the means of expression as usually adopted in non-formalized proofs; just in this sense his formalization is *more natural.* Whether the word '*consider*' or another word is to be used for the reading of 'T', this is a stylistic question which is irrelevant for the present discussion. Let it only be noted that the phrase '*let x be so-and-so*' may function as well to hint at the elimination of the universal quantifier, while '*consider*' in the context '*consider x satisfying ϕ*' seems to better express the existential quantifier elimination.

3. The Issue of Naturalness Continued: A Lesson of Euclid

It is worth examining the strategy of dealing with quantifiers as it appears in the paradigm of mathematical arguments, that is in *The Elements of Euclid* (quotations after I. Todhunter's edition, re-edited by E. Rhys, London 1933). Let us take Proposition 1 in Book I. It states the following problem: *To describe an equilateral triangle on a given finite straight line.* This amounts to the statement prefixed with two quantifiers, viz.: for *any* finite straight line *there exists* an equilateral triangle that can be described on this line. Note that the nouns such as 'line' can be naturally interpreted as variables ranging over a universe in a multi-sorted system, like that of Hilbert (1899) in his *Grundlagen der Geometrie.* Then our proposition can be translated into the language of multi-sorted predicate logic in the following way: *For any line x there is a plane figure y such that: y is an equilateral triangle, and y is described on x.*

In the proof of Proposition 1, which will be called 'the thesis' below, the first sentence starts from the following assumption: *Let AB be the given line.* It is remarkable that the phrase 'a given line', that means in the thesis any line whatever, switches to the phrase '*the given* line' in this starting assumption. This is a clear case of eliminating the existential quantifier in favour of an indefinite term (or, an undefined constant, in

Jaśkowski terminology). In Euclid this operation occurs without any explicit reference to the rule of quantifier elimination, while the use of the phrase starting from 'let' can be duly regarded as an implicit reference to this intuitive and general law of reasoning. In Jaśkowski's system the same role is played by inserting the symbol 'T' inside the relevant proof line.

Now we can see degrees of naturalness in formalizing a proof. Jaśkowski's procedure is closer to that of Euclid and other mathematicians than the procedure, say, of the SB system; in this sense it becomes more natural. Then, if somebody provides us with such a formalization in which natural language expressions like '*let*' (instead of Jaśkowski's 'T') and '*assume*' (Jaśkowski's 'S') are employed, this formalization should be acknowledged as still more natural than that of Jaśkowski. Such a highly natural formalization, at the same time being fully suited for a computer, is the main subject of the present discussion; it is to be presented in the next sections.

It is useful for the forthcoming discussion to look at some other facts in Euclid. In the proof in question, the quoted assumption 'Let AB be the given straight line' is followed by the problem statement that can be read as an existential statement since the problem consists in producing a required object, thus proving its existence; this runs as follows (after the colon): 'it is required to describe an equilateral triangle on AB'. The article prefixing 'triangle' is again indefinite, as it was in the thesis, and thus the existential meaning remains preserved. Then the required triangle gets constructed and obtains the name 'ABC'. After it is constructed and so termed, its name deserves to be preceded with the definite article in the conclusion which reads as follows: 'Wherefore *the triangle ABC is equilateral, and it is described on the given straight line AB*.' Now both indefinite articles in the thesis in question, as stated at the start, are replaced with the definite articles, one before 'line' and one before 'triangle'.

In such a statement of the conclusion, again some points regarding naturalness deserve a comment. A minor point consists in using Roman fonts for the word 'wherefore', and italics for the conclusion itself; this typographical strategy, following that of Todhunter's edition, helps to distinguish between the content of the proof and such text junctures as the words 'wherefore', 'because', 'but' appearing in the analyzed proof. A formalization of mathematical language that pretends to be as natural as possible should provide us with such vernacular junctures instead of,

e.g., the symbol $\vdash$ (to be read 'wherefore').

The major point is concerned with the rule of generalization. Again, there is no explicit reference to this rule in the proof conclusion. But an implicit reference can be seen in the use of the definite article before 'line'. The history of dealing with the phrase 'the line' in the course of the proof in question is sufficient reason to read 'the line' as 'every line'; the main part of this plot consists of using 'let' in the moment of passing from the indefinite to the definite article. Nothing of this kind happened to the predicate 'triangle', and this is why the meaning of 'the' as prefixing this predicate in the conclusion has to be different from the meaning of 'the' in the phrase 'the line'. The first meaning, that attached above to 'the line', can be duly called *platonic* for its frequent appearances in Plato, e.g. in *The Republic*, where 'the' hints at an individual object taken as standing for a whole class; with the Schoolmen, this kind of use was called *suppositio formalis*. The second meaning of 'the', in our example attached to 'triangle', resembles the *suppositio personalis* of medieval logicians, and applies to individuals as individuals (not as substitutes for classes).

These considerations lead to the next postulate addressed to any formalization trying to be as natural as possible: it should use generalization in a way similar to that appearing in Euclid, that also in this respect has created the literary paradigm, i.e. text composition rules, for the whole of mathematics. We shall see that also in this respect the system to be presented matches those high requirements, and succeeds in adopting computers for assistance in natural inferences.

PART TWO: ON MIZAR MSE

4. *The Name and What It Stands For*

Do not try to guess what the name "Mizar" means. It was the author's fancy to take a star's name to stand for (i) a natural deduction system of (ii) Multi-Sorted predicate logic with Equality, for short MSE, (iii) that simulates the language of proofs, especially that used by mathematicians, in a simplified and standardized form, adjusted to computer processing, and (iv) that is combined with a *checker*, i.e. a program checking proof validity (checkers should be distinguished from *provers*; for the latter see Bläsius and Bürckers 1987). Logical foundations for constructing provers and checkers have been prepared for by such the-

oretical results as those of Gentzen (1934–1935), Herbrand (1930), Beth (1955); an instructive presentation of these achievements and their computer-oriented continuation is found in Robinson (1979).

Thus Mizar MSE, later on referred to as MSE, belongs to the class of systems devised for Computer-Assisted Reasoning, CAR for short. Both the logical system and the corresponding pieces of software are due to Andrzej Trybulec and his team in the Institute of Mathematics, Warsaw University, Białystok Branch, while the Logic Department of the same University functions as a laboratory for testing the system's fitness in teaching predicate logic.

What distinguished MSE from some other CAR systems is that it offers various modes of proof construction regarding text structure and vocabulary. This gives MSE linguistic flexibility, a feature welcomed by many users. What distinguishes MSE from traditional formalizations is that only premises are explicitly referred to in justifying an inference, while inference rules, like in natural mathematical arguments, implicitly appear in a verbal context of symbolic expressions; e.g. "consider x such that" which, when preceding a formula, corresponds to the rule of existential quantifier elimination (the choice rule). Owing to these and other solutions in the Mizar language, the proofs written in it are more like ordinary mathematical texts than long and cumbersome formalizations in the Hilbert program style (see Hilbert and Bernays 1934). The price for these conveniences is that a richer vocabulary has to be mastered than that of ordinary predicate logic.

The MSE system presented in these Comments does not pretend to suit the whole practice of reasoning in mathematics. Its use is restricted to elementary theories without functions. Note however, that multi-sortedness extends the possibilities of expression by introducing abstract objects as, so to say, abstract individuals, viz. sets, properties, relations, etc.

The presentation offered below is concerned with that version of MSE which is used together with the specially devised editing program. Some references to the technicalities of this editor appearing below may seem too detailed, however, they are meant to give an idea of how the MSE program works.

5. On What Is Obvious in Proving

Any simulation of human reasoning involves the decision about what kinds of inferences are to be regarded as obvious. This is an important philosophical point. The contrary philosophical claim would be that of Descartes who seems to have believed that there are the smallest inferential steps, like atoms of logical evidence, which are shared by all human reasoners. A similar belief in atoms of evidence, though the atoms themselves are conceived of very differently, is found in Hilbert and Bernays' method of formalization. If a checker constructor does not acknowledge such a claim, then he takes advantages of freedom of choice among various admissible kinds of evidence.

The above philosophical point is crucial for understanding certain MSE features. This system arises from conscious decisions regarding the criteria of obviousness to be offered to system users. These criteria can be replaced with other ones, according to the user's supposed needs or the author's varying ideas. The user soon recognizes these features of MSE when at the very start he encounters a non-conventional approach. Contrary to textbook standards, MSE acknowledges all truth-functional tautologies as being obvious, independently of a degree of syntactic complexity. Furthermore, some predicate logic tautologies are regarded as obvious (see examples); in this point MSE differs from other formalized inferential systems, those in which every step should be justified by an explicit appeal to inference rules.

The fact that a formula is treated as obvious can in MSE be discovered in an experimental way. Namely, if the system gives its OK to a formula under proof without any intermediary steps, i.e. when one writes it down directly in the line following the title *proof* and precedes it with *thus*, then the inference is to be recognized as obvious. This can be seen in Example 5:1.

```
environ
reserve x for integer;
A: not (for x holds P[x]);
   :: In the usual notation: ∼ ∀xP(x))
```

```
begin
T: ex x st (not P[x])
   :: In the usual notation: ∃x ∼ P(x);
   :: In Mizar we use 'for' as short for 'for every', and 'st' as short
for 'satisfying'
proof
thus thesis by A;
end;
```

Apart from parentheses, two punctuation devices are adopted: the colon separates the name of a formula from the formula in question, the semicolon ends every proof line (except for the thesis to be proven) and the whole proof. Double colons separate the proof itself, as controlled by the checker, from comments which may be added for the user's convenience and are disregarded by the checker. The item *given* used in the premise forms a specially devised context to hint at an individual constant. MSE allows any letter string to be taken for a constant, and also any letter string for a variable, for these different functions are distinguishable owing to different contexts in MSE; however, for a better readability let a, b, c, a1, Socrates, etc. stand below for constants, and let x, y, z, x1, etc. stand for variables. This is illustrated in Example 5:2.

```
environ
reserve x, Wałesa for human;
given Wałesa being human;
A: nobelprizewinner[Wałesa];
begin
T: ex x st nobelprizewinner[x]
proof
thus thesis by A; end;
```

Note, if we are not interested in specifying a particular sort, we can declare it as anything, or even use a dummy term, for instance 't'; the lack of such a type-specifying clause is treated in MSE as an error.

A kind of obviousness attaches to the axioms of identity, in the sense that they are 'tacitly" built into the environ of each proof, and can be used without any referencing. There is a fairly wide literature on the notion of obviousness as relativized to various checkers; some data on this subject are found in Rudnicki (1987).

6. *'Assume', 'Let', 'Given x Such That'*

As in the other inferential systems, when proving a conditional in MSE we assume the antecedent (hypothesis), or its part. This step is marked with the item *assume*; it does not require *justification*, i.e. referencing an (earlier proven) premise, for we may take advantage of a false hypothesis as well. (Note, what is to be proven is not the consequent but the whole conditional, and this can be true while its antecedent happens to be false.)

The item *let*, specific for MSE, is the counterpart of the combined use both of elimination and of introduction of the universal quantifier, the latter usually being called *generalization*. The *let* statement is used to pick up and to fix a random object as representing all objects of a certain type. This object is denoted with a constant that will be generalized in a next step, the constant being included within the scope of *for* (the universal quantifier). Both *assume* and *let* constructions are shown in Example 6:1.

```
environ
reserve a, y for t;
begin
T1: (for y holds P[y]) implies (ex y st P[y])
proof
assume 1: for y holds P[y];        ::   1 · S∀yP(y)
now let a be t;                    ::   1 · 1Ta
P[a] by 1;                         ::   1 · 1P(a)
hence ex y st P[y];                ::   1 · ∃yP(y)
end;
hence thesis; end;                 ::   ∀yP(y)
```

This example is to illustrate several points. There is a historical point hinted after the double colons (used to separate comments which are disregarded by the checker). The numbering follows Jaśkowski's method as exemplified in Section 2 above to show the analogy to the MSE approach. 'S' means 'suppose' (in MSE '*assume*'; '1.1' is to introduce nesting (1.1 as nested in 1) which in MSE is rendered with '*now*' while 'T' serves the purpose of introducing a fixed random term, and this is just that what '*let*' does in MSE.

As mentioned above, this text in a simple way exemplifies a nesting whose beginning is marked with '*now*' and the end with its own '*end*'.

The junction word '*hence*' has a role similar to that of '*thus*', with the difference that it prefixes the statement which is justified by the immediately preceding line, and therefore the justifying reference ('*by*', etc.) is omitted. Such stylistic variations derive from MSE's philosophy that natural modes of expressions in reasonings should be imitated in computer formalization.

When the antecedent of a conditional is an existential proposition, then the assumption has to be accompanied by the existential quantifier elimination. This is done with the item '*given a such that*' (not to be mistaken for the construction '*given a being*' from Example 5.2). Let this be shown in Example 6.2.

```
environ
reserve a, x for t;
begin
T: (ex x st (p[x] & q[ ])) implies ((ex x st p[x]) & (ex x st q[ ]))
proof
given a such that 1: p[a] & q[ ];
thus thesis by 1;
end;
```

This example gives us the opportunity to observe that our editor permits the use of lower case instead of capitals in the role of predicates; at the same time it always requires brackets to follow predicates, even if no arguments are to be enclosed. The same example shows the obviousness of truth-functional tautologies and of existential generalization, as commented on above, in Section 2. To obtain our thesis from 1, we need to split 1 into its two members (according to the truth-functional law of simplification), then to apply existential generalization (cf. Example 2) to each member, and then to join them according to the rule of introducing conjunction. None of these steps needs to be mentioned, as all of them are treated as obvious.

7. 'Consider' (The Choice Rule)

If it is stated that an object satisfying certain conditions exists, then it is permissible to introduce such an object giving it a name. This principle is called the choice rule. When it is employed in a MSE text, its use is expressed with a *consider* statement including a constant, as it can be seen in Example 7:1.

```
environ
reserve a, x for t;
begin
T: (ex x st p[x] & q[x]) implies ((ex x st p[x]) & (ex x st q[x]))
proof
assume A: ex x st p[x] & q[x];
B1: ex x st p[x]
proof
consider a such that A1: p[a] & q[a] by A;
B2: p[a] by A1;
thus thesis by B2;
end;
C1: ex x st q[x]
proof
consider a such that A1: p[a] & q[a] by A;
C2: q[a] by A1;
thus thesis by C2;
end;
thus thesis by B1, C1;
end;
```

Besides the use of *consider*, this example shows that sometimes an auxiliary proof must be placed inside a larger argument; the former is what we call a nested proof. Nested proofs can be distinguished from the main proof by indenting, which makes them more readable; however, this device is not demanded by MSE. In the second nested proof (that of thesis C1) one could have used a different name, say ‘b’, than the one used earlier (‘a’), provided ‘b’ has been reserved for this purpose in the type declaration in the environ. A comparison of *consider* with *let* will be in order. The *consider* statement requires a justification while the *let* statement does not. It is so because the former is being asserted whereas the latter only supposed, and whether the supposition is true or false does not affect the truth or the falsity of the generalization to be proven. On the other hand, the application of the choice rule depends on the actual existence of the object being introduced, and therefore the relevant existential statement should be mentioned in the justification clause.

A useful application of *consider* is found in the indirect proofs of universal propositions and also of those conditionals which have a uni-

versal proposition in the consequent. Let the latter be exemplified with the following story – Example 7:2.

```
environ
reserve x, Robin Hood for bold;
A: for x holds (danger[ ] implies happy[x]);
begin
T: danger[ ] implies (for x holds happy[x])
proof
assume 1: not thesis;
2: danger[ ] by 1;
3: not (for x holds happy[x]) by 1;
consider Robin-Hood such that 4: not happy[Robin-Hood] by 3;
5: danger[ ] implies happy[Robin-Hood] by A;
6: happy[Robin-Hood] by 2, 5;
thus contradiction by 4, 6;
end;
```

Obviously, 'danger' is short for 'there-is-a-danger', and 'happy' for 'is-happy' (if written with such dashes, the phrases will be accepted by MSE's syntactic rules). Thus fairyland reasonings can be told in MSE as well, so permissive is its syntax. It proves even more permissive due to some stylistic variants, some of them being meant as shortcuts.

Let the following be listed. Parentheses in A and T can be omitted. Premise A can be stated in a shorter form, resembling restricted quantification, viz.: 'for x st danger[ ] holds happy[x]' (though this stylistic version does not seem lucky for this particular example). The conclusion can also run as follows: 'hence contradiction by 4' which is to mean that line 6 is tacitly taken into account as the immediately preceding one, and therefore only the more remote line 4 is referenced. These are samples of MSE's linguistic flexibility which can be more and more extended, according to users' tastes.

8. Concluding Remarks

Finally, let it be observed that MSE is not only a piece of software devised as a checker for an earlier existing system of predicate logic, as is the case, e.g., with TABLEAUX II. The latter, produced by Duncan Watt, Oxford University, functions like a software appendix to the system of analytic tableaux, especially that found in Smullyan (1968)

which lies in the Gentzen-Beth tradition. MSE is a system which contains a software integrated with a new logical language and a set of inference rules. The latter has some resemblances to Jaśkowski's approach but in some important points, such as multi-sortedness and the way of dealing with generalization, brings original solutions of its own. That such theoretical solutions in logic have been combined with a suitable software, gives this system useful applications in teaching predicate logic; see e.g. A. Trybulec and Blair (1985 and 1985a), Mostowski and Z. Trybulec (1985), Zalewska (1987), Marciszewski (1987), Artalejo (1988), Malinowski (1990), Mostowski (1991).

A no less important field of educational application is found in teaching mathematics. This should be compared with the most advanced projects in this field, such as that of Patrick Suppes at Stanford, reported in Suppes (1981); see also Marinov (1983), Mizar applications in the computer-assisted teaching of mathematics are discussed, e.g. in Rudnicki (1982); Szczerba (1987). Mizar can also be applied to software verification; see, e.g. Rudnicki and Drabent (1985).

A much wider range of applications attaches to the system called PC Mizar, devised by the same author Andrzej Trybulec supported by a number of collaborators from Polish and foreign universities. Its main ideas, including those related to Jaśkowski, are exemplified in Mizar MSE, but this full Mizar system offers more powerful tools, especially for dealing with functions in mathematical proofs, so that the whole of mathematics can be conveniently expressed in it. At the level of editing texts, a crucial difference between MSE and PC Mizar consists in the system of references functioning in the latter. While in MSE for each proof its premises have to be listed in the *environ* section, in PC Mizar one can take advantage of either axioms or proven theorems mentioned in other Mizar articles belonging to the Mizar Library.[1]

Obviously, such a project demands international collaboration. It has got more chances since the journal *Formalized Mathematics: A Computer Assisted Approach* appears every two months, edited by Roman Matuszewski. Such a library should be a suitable basis for an international mathematical data bank, and this is a prospective enterprise. Also in this way some Polish mathematicians and logicians try to continue the traditions of both disciplines in their country. Some news on the Mizar Library is found in Rudnicki and Trybulec (1989), more recent in A. Trybulec's Introduction to the first issue of the mentioned *Formalized Mathematics* (1990).

Let it be added that a system aspiring to a similar universality is being developed by a very competent team in Eindhoven, the Netherlands, under the name AUTOMATH. That system, unlike Mizar, is not based on classical predicate logic but on lambda calculus (see de Bruijn, 1980, 1985). At the moment one cannot tell whether the Mizar approach or the Automath approach, or any other one will succeed in becoming a standard computer system for a worldwide net of collaboration in mathematics. Nevertheless, it seems certain that such a system should be created as a desirable device for mathematical knowledge management.

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NOTE

[1] To order copies of this journal as well as copies of the Mizar software, address Roman Matuszewski, e-mail romat at plearn.

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ROMAN MURAWSKI

HILBERT'S PROGRAM: INCOMPLETENESS THEOREMS VS. PARTIAL REALIZATIONS

1. HILBERT'S PROGRAM

Mathematics on the turn of the 19th century was characterized by the intense development on the one hand and by the appearance of some difficulties in its foundations on the other. Main controversy centured around the problem of the legitimacy of abstract objects. The works of K. Weierstrass have contributed to the clarification of the role of the infinite in calculus. Set theory founded and developed by G. Cantor promised to mathematics new heights of generality, clarity and rigor. Unfortunately paradoxes appeared. Some of them were known already to Cantor (e.g. the paradox of the set of all ordinals and the paradox of the set of all sets[1]) and they could be removed by appropriate modifications of set theory (cf. Cantor's distinction between *absolut unendliche* or *inkonsistente Vielheiten* and *konsistente Vielheiten*, i.e. between classes and sets[2]). Frege's attempt to realize the idea of the reduction of mathematics to logic (which was in fact a continuation of the idea of the arithmetization of analysis developed among others by Weierstrass) led to a really embarrasing contradiction discovered in Frege's system by B. Russell and known today as Russell's antinomy or as the antinomy of non-reflexive classes. This meant a crisis of the foundations of
mathematics (called the second crisis – the first being the crisis caused by the discovery of incommensurable segments in the ancient Greek mathematics).

Various ways of overcoming those difficulties and of securing the edifice of mathematics were proposed. Great mathematicians – e.g. L. Kronecker, H. Poincaré, L. E. J. Brouwer, H. Weyl – challenged
the validity of all infinitistic reasonings and proposed to restrict methods of mathematics to secure finite ones. L. Kronecker rejected any infinite objects restricting mathematics to integers only ("Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk" – as

Jan Woleński (ed.), Philosophical Logic in Poland, 103–127.

he formulated his scientific and methodological credo during a meeting in Berlin in 1886). H. Poincaré saw the source of antinomies in impredicativity of mathematics and demanded a restriction to predicative methods only. The radical proposal of L. E. J. Brouwer, known today as intuitionism, was based on the idea that mathematics should be founded on the primitive intuition of natural number. He claimed that mathematics is a free activity of human mind, it can (and should) be developed independently of any language, one should restrict only to constructive methods, hence in particular any non-constructive proofs of existential sentences should be rejected (Brouwer claimed that proofs of that type were the source of all antinomies). He accepted only countable infinity but rejected any uncountable one.

All those proposals meant in fact a restriction of mathematics and a rejection of a great part of it, especially that dealing with infinite objects. D. Hilbert was definitely against it. He wrote: "What Weyl and Brouwer do comes to the same thing as to follow in the footsteps of Kronecker! They seek to save mathematics by throwing overboard all that which is troublesome ... They should chop up and mangle the science. If we would follow such a reform as the one they suggest, we would run the risk of losing a great part of our most valuable treasures!" (Reid, 1970, p. 155). And added: "I believe that as little as Kronecker was able to abolish the irrational numbers ... just as little will Weyl and Brouwer today be able to succeed. Brouwer is not, as Weyl believes him to be, the Revolution – only the repetition of an attempted Putch" (Reid, 1970, p. 157). And he stressed firmly that: "Aus dem Paradis, das Cantor uns geschaffen hat, soll uns niemand vertreiben können" (cf. Hilbert, 1926).

Hilbert proposed a method of justification of (infinite) mathematics known today as Hilbert's program. It was the core of a new doctrine in the philosophy of mathematics called formalism (which became one of the main trends of the modern philosophy of mathematics beside Frege's and Russell's logicism and Brouwer's intuitionism).

Hilbert was first of all a mathematician and – as Smoryński writes – "had little patience with philosophy, his own philosophy of mathematics being perhaps best described as naïve optimism – a faith in the mathematician's ability to solve any problem he might set for himself" (cf. Smoryński, 1988). His aim was to save the integrity of classical mathematics (dealing with actual infinity) by showing that it is secure.[3] This problem was first stated by him in his lecture at the Second Inter-

national Congress of Mathematicians held in Paris in 1900 (cf. Hilbert, 1901), then repeated in a number of articles in the twenties where he proposed a method of solving it (a good account of the development of Hilbert's views can be found in Smoryński (1988); see also Peckhaus (1990) where detailed analysis of Hilbert's scientific activity in the field of foundations of mathematics in the period 1899–1917 can be found). Hilbert saw the supramathematical significance of this issue. He wrote in Hilbert (1926): "The definitive clarification of the nature of the infinite has become necessary, not merely for the special interests of the individual sciences but for the honor of human understanding itself."

Hilbert's program of clarification and justification of mathematics was kantian in character. Following Kant he claimed that the mathematician's infinity does not correspond to anything in the physical world, that it is "an idea of pure reason" – as Kant used to say. On the other hand:

> Kant taught – and it is an integral part of his doctrine – that mathematics treats a subject matter which is given independently of logic. Mathematics, therefore can never be grounded solely on logic. Consequently, Frege's and Dedekind's attempts to so ground it were doomed to failure.
>
> As a further precondition for using logical deduction and carrying out logical operations, something must be given in conception, viz., certain extralogical concrete objects which are intuited as directly experienced prior to all thinking. For logical deduction to be certain, we must be able to see every aspect of these objects, and their properties, differences, sequences, and contiguities must be given, together with the objects themselves, as something which cannot be reduced to something else and which requires no reduction. This is the basic philosophy which I find necessary not just for mathematics, but for all scientific thinking, understanding and communicating. The subject matter of mathematics is, in accordance with this theory, the concrete symbols themselves whose structure is immediately clear and recognizable. (Cf. Hilbert, 1926).

According to this Hilbert distinguished between the unproblematic, 'finitistic' part of mathematics and the 'infinitistic' part which needed justification. Finitistic mathematics deals with so called real sentences, which are completely meaningful because they refer only to given concrete objects. Infinitistic mathematics on the other hand deals with so called ideal sentences that contain reference to infinite totalities. Hilbert believed that every true finitary proposition had a finitary proof. Infinitistic objects and methods played only an auxiliary role. They enabled us to give easier, shorter and more elegant proofs but every such proof could be replaced by a finitary one. He was convinced that consistency implies existence and that every proof of existence not giving a construction of postulated objects is in fact a presage of such a construction. (Compare in connection with this Hilbert's solution in 1888 to Gordan's

problem in the theory of invariants in which he proved, without construction, the existence of a finite base for any ideal in the polynomial ring $K[X_0, ...X_{n-1}]$ over a field K.)

Unfortunately Hilbert did not give a precise definition of finitism – one finds by him only some hints how to understand it. Hence various interpretations are possible. Usually it is assumed that a finitist reasoning is essentially a primitive recursive reasoning in the sense of Skolem (cf. Tait, 1981; Resnik, 1974). But there are also other interpretations – cf. e.g. Detlefsen (1979) where it is suggested that even some variants of ω-rule can be regarded as finitistic or Smoryński (1988) where instead of a dichotomy real/ideal a trichotomy real/finitary general/ideal is proposed (cf. also the critisism of this proposal in Detlefsen (1990)). Prawitz argues in Prawitz (1981) that real sentences are the decidable ones (i.e. numerical equations and truth-functional compositions of them) and the ones of the form $\forall x A(x)$ where each instance $A(t)$ is decidable. The rest are considered to be ideal. This emphasizes the role of Π_1^o sentences in Hilbert's program (cf. Kitcher, 1976; Tait, 1981).

The infinitistic mathematics can be justified only by finitistic methods because only they can give it security (*Sicherheit*). Hilbert's proposal was to base mathematics on finitistic mathematics via proof theory. Its main goal was to show that proofs which use ideal elements in order to prove results in the real part of mathematics always yield correct results. One can distinguish here two aspects: consistency problem and conservation problem. In some of Hilbert's publications (cf. e.g. Hilbert, 1926; Hilbert, 1927) both aspects are stressed but usually (cf. e.g. Hilbert's last publication on this subject, namely the first volume of Hilbert and Bernays, 1934–1939) the one-sided emphasis is put on the consistency problem only (cf. also Kreisel (1968) and Kreisel (1976) where the author calls for a proper formulation taking into account both aspects). The consistency problem consists in showing (by finitistic methods, of course) that the infinitistic mathematics is consistent; the conservation problem consists in showing by finitistic methods that any real sentence which can be proved in the infinitistic part of mathematics can be proved also in the finitistic part, i.e. that infinitistic mathematics is conservative over finitistic mathematics with respect to real sentences and, even more, that there is a finitistic method of translating infinitistic proofs of real sentences into finitistic ones. Both those aspects are interconnected (what was indicated by Kreisel – we shall discuss this problem later).

Hilbert's proposal to carry out this program consisted of two steps. The first step was to formalize mathematics, i.e. to reconsitute infinitistic mathematics as a big, elaborate formal system (containing classical logic, infinite set theory, arithmetic of natural numbers, analysis). An artificial symbolic language and rules of building well-formed formulas should be fixed. Next axioms and rules of inference (referring only to the form, to the shape of formulas and not to their sense or meaning) ought to be introduced. In such a way theorems of mathematics become those formulas of the formal language which have a formal proof based on a given set of axioms and given rules of inference. There was one condition put on the set of axioms (and rules of inference): they ought to be chosen in such a way that they suffice to solve any problem formulated in the language of the considered theory as a real sentence, i.e. they ought to form a complete set of axioms with respect to real sentences.

The second step of Hilbert's program was to give a proof of the consistency and conservativeness of mathematics. Such a proof should be carried out by finitistic methods. This was possible since the formulas of the system of formalized mathematics are strings of symbols and proofs are strings of formulas. i.e. strings of strings of symbols. As such they can be manipulated finitistically. To prove the consistency it suffices to show that there are not two sequences of formulas (two formal proofs) such that one of them has as its end element a formula φ and the other $\neg\varphi$ (the negation of the formula φ). To show conservativeness it should be proved that any proof of a real sentence can be transformed into a proof not referring to ideal objects.

One should note here that formalization was for Hilbert only an instrument used to prove the correctness of (infinitistic) mathematics. Hence the objections raised to him by Brouwer are mistaken. As indicated in (Kreisel, 1964) the real opposition between Brouwer's and Hilbert's approach to mathematics was between: (i) the conception of what constitutes a foundation and (ii) two informal ways of reasoning, namely finitist and intuitionist. Recall that Brouwer ignored non-constructive mathematics altogether.

Note also that if one identifies real sentences with Π^0_1 sentences (see above) then – as shown by Kreisel – a solution to the consistency problem yields a solution to the conservation problem (Kreisel's results are presented e.g. in Smoryński (1977), pp. 858–860).

2. INCOMPLETENESS RESULTS

Hilbert and his school had scored some successes in realization of the program of justifying infinite mathematics. In particular Hilbert's student W. Ackermann showed by finitistic methods the consistency of a fragment of arithmetic of natural numbers (cf. Ackermann, 1924–1925 and Ackermann, 1940). But soon something was to happen that undermined Hilbert's program.

In September 1930 a Conference on Epistemology of Exact Sciences (organized by Gesellschaft für empirische Philosophie) was held in Königsberg. On September 7th (the last day of the conference) a young Austrian mathematician Kurt Gödel presented a short announcement in which he reported on his recent result on incompleteness of arithmetic of natural numbers. The result known today as Gödel's first incompleteness theorem was published in January 1931 in a paper "Über formal unentscheidbare Sätze der 'Principia Mathematica' und verwandter Systeme. I" (cf. Gödel, 1931). It states, roughly speaking, that arithmetic of natural numbers and all systems containing it are essentially incomplete provided they are consistent. It means that there are sentences which are undecidable in them, i.e. sentences φ such that neither φ, nor $\neg\varphi$ are theorems. What more one knows which sentence of the pair $\varphi, \neg\varphi$ is true (in the basic model of the theory, i.e. in the model to the description of which the theory was formulated). This incompleteness is essential, i.e. it cannot be removed by adding the undecidable sentences as new axioms because new undecidable sentences will appear (undecidable in the new stronger theory).

Hence Gödel's theorem indicated certain cognitive limitations of the deductive method. It showed that one cannot include the whole mathematics in a consistent formalized system based on the first order predicate calculus – what more, one cannot even include in such a system all truths about natural numbers. There will be always, as Gödel proved, undecidable sentences of the form $\forall x\varphi(x)$ such that all instances of φ, i.e. sentences $\varphi(0), \varphi(1), \varphi(2), \ldots$ are theorems. Gödel's undecidable sentence constructed by the diagonal method stated its own unprovability ("I am not a theorem") (the construction of such a sentence was possible thanks to Gödel's idea of arithmetization of syntax). On the other hand one can prove, using some infinitistic methods (model-theoretical or set-theoretical ones) that this sentence is true. Hence we have an example of a (real) sentence (referring to natural numbers only)

which can be proved by some infinitistic methods but which has no arithmetical proof.

Gödel's paper contained also at the end an announcement (with a promise to give a proof in the second part of the paper which in fact was never written[4]) of another theorem, called today Gödel's second incompleteness theorem and stating that no formal theory containing arithmetic of natural numbers can prove its own consistency.

Note that Gödel's remark (in the paper from 1931) that one can prove the second incompleteness theorem by formalizing the proof of the first incompleteness theorem was not correct. The first full proof of unprovability of consistency was given in Hilbert and Bernays (1934–1939). It has turned out that the way of formalizing the metamathematical sentence "the theory T is consistent" in the formal language of T is significant. Hilbert and Bernays formulated certain so called derivability conditions for formulas representing in T the metamathematical notion of provability (in fact those conditions require certain internal properties of provability to be formally derivable in T). If those conditions are fulfilled then the second incompleteness theorem holds. M. H. Löb gave in 1954 another, more elegant form of derivability conditions (cf. Löb, 1955). It was proved also that there exist formal translations of the sentence "T is consistent" which are provable in T (hence for them Gödel's second theorem fails). An example of such a formula was given in Rosser (1936). A detailed analysis of this problem can be found in Feferman (1960).

Gödel's results struck Hilbert's program. Did they reject it? This question cannot be answered definitely for the simple reason – Hilbert's program was not formulated precisely enough, it used vague terms as finitistic, real, ideal which were never precisely defined. In this situation various opinions are formulated and defended. On the one hand it is claimed that Gödel's incompleteness results showed the failure of Hilbert's program – cf. e.g. Smoryński (1977, 1985, 1988). On the other it is argued that it is not the case. Several reasons are given here. For example, Detlefsen (1979) says that the second Gödel's theorem does not imply the rejection of Hilbert's proposal bacause the unprovable formal sentence stating the consistency of the theory "does nor really 'express' consistency" in the sense meant by Hilbert. In Detlefsen (1990) it is argued that Gödel's first incompleteness theorem does not refute the program because Hilbert did not demand the conservation property but only a weak conservation, i.e. conservation

with respect to real sentences which can be finitistically decided and he did not claim that every real sentence may be decided in such a way.[5]

As indicated above, Gödel's second incompleteness theorem requires certain assumptions about the formal translation of some metamathematical notions (such as 'proof', 'provability', etc.). Detlefsen observes in Detlefsen (1990) that they are not satified by various theories or 'theory-like' arrangements of proofs and theorems which nonetheless do satisfy the conditions required by Gödel's first theorem.[6] He considers so called 'consistency-minded' theories which incorporate consistency constraints into the very conditions on proof, provability etc. Two types of such theories are distinguished: Rosser systems (studied in Rosser, 1936; Kreisel and Takeuti, 1974; Guaspari and Solovay, 1979; Visser, 1989; Arai, 1990) and Feferman-systems (introduced in Feferman, 1960 and studied in Jeroslov, 1975; Visser, 1989). Since some of those theories constitute plausible ways in which the Hilbertian might go about constructing his ideal theories, Detlefsen formulates the following open question[7]: "whether G2 [i.e. Gödel's incompleteness theorem – R.M.] applies to Hilbert's Program *per se*, or only to those versions of it which needlessly restrict themselves to theory construction of the usual static variety" (p. 345).

Resnik argues in Resnik (1974) that the incompleteness theorem "has less bearing upon the [Hilbert's] programme than is often credited to it!" This thesis is based on the claim that: "every formal system to which Gödel's theorems apply is complete with respect to its real sentences. Thus the undecidable sentences are ideal sentences."

Note also that the failure of Hilbert's program for a certain formalized system of arithmetic need not be a failure of Hilbert's program for elementary number theory in the informal sense. In fact one cannot exclude the possibilitty that the latter can be formalized in a system which can be justified on finitistic grounds.

And what were the reactions and opinions of the main heros of the whole story, that is of Hilbert and Gödel? Hilbert, though taking part in the conference in Königsberg, did not learn about Gödel's result. He did it only by the early part of 1931. As Smoryński writes: "He was angry at first, but was soon trying to find a way around it" (Smoryński, 1988). He proposed to add to the rules of inference a simple form of the ω-rule (which allows the derivation of all true arithmetic sentences). In Preface to the first volume of Hilbert and Bernays (1934–1939) Hilbert wrote:

... the occasionally held opinion, that from the results of Gödel follows the non-executability of my Proof Theory, is shown to be erroneous. This result shows indeed only that for more advanced consistency proofs one must use the finite standpoint in a deeper way than is necessary for the consideration of elementary formalisms.

K. Gödel in his 1931 paper wrote:

I wish to note expressly that Theorem XI (and the corresponding results for M and A) do not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used, and it is conceivable that there exist finitary proofs that cannot be expressed in the formalism of P(or M or A).[8]

(English translation taken from van Heijenoort (1967), p. 615).

At the Vienna Circle meeting on January 15, 1931 Gödel argued that it is doubtful, "whether all intuitionistically correct proofs can be captured in a *single* formal system. That is the weak spot in Neumann's argumentation" (quotation taken from Sieg, 1988). In (Gödel, 1946) he explicitly called for an effort to use progressively more powerful transfinite theories to derive new arithmetical theorems.

3. GENERALIZATIONS AND STRENGTHENINGS OF GODEL'S THEOREMS

Beside all the discussions and doubts to connections between Gödel's incompleteness theorems and Hilbert's program described above one question more should be raised.

Gödel in his first incompleteness theorem indicated a true arithmetic sentence which is undecidable in the given formal system of arithmetic. This sentence has not a mathematical but in fact a metamathematical contents (it states: "I am not a theorem"). This diminished the meaning and significance of Gödel's theorem. Also other undecidable sentences constructed later (e.g. sentences of Rosser, Kreisel and Levy, Kent, Mostowski, Shepherdson – see Smoryński, 1981) have this failure. Though interesting for logicians they are rather artificial from the mathematical point of view. Hence an open question arose: is it possible to indicate examples of undecidable sentences of mathematical, in particular number-theoretical, contents? This question was even more interesting because it was still possible to charish hopes that all sentences which are interesting and reasonable from the mathematical point of view (whatever it means) are decidable.

This problem was solved in 1977 – it was done by J. Paris (cf. Paris, 1978). Working on non-standard models of Peano arithmetic

PA he has invented a new method of constructing sentences which are independent of PA but true (in the standard model $\mathfrak{N}_0$). The sentences of Paris were simplified by L. Harrington and at the end a new elegant undecidable sentence of a combinatorial contents was obtained (cf. Paris and Harrington, 1977).

To describe Paris-Harrington sentence we need to fix some notation. By Peano arithmetic PA (which is now a standard formal system used in studies of the foundations of mathematics) we mean a first order theory based on the classical predicate calculus and on the following non-logical axioms:

$$Sx = Sy \to x = y$$

$$Sx \neq 0$$

$$x + 0 = x$$

$$x + Sy = S(x + y)$$

$$x \cdot 0 = 0$$

$$x \cdot Sy = x \cdot y + x$$

$$\varphi(0) \,\&\, \forall x[\varphi(x) \to \varphi(Sx)] \to \forall x \varphi(x),$$

where $\varphi(x)$ is any formula of the language of PA with the free variable x (and possibly some other free variables treated as parameters). The standard model $\mathfrak{N}_0$ of PA is the structure $< N, S, +, \cdot, 0 >$ where N is the set of natural numbers, S is the successor function, + and $\cdot$ are addition and multiplication, respectively, and 0 denotes the number zero.

If X is a set of natural numbers then $[X]^n$ denotes the family of all n-element subsets of X. A function $C : [X]^n \to c$ (c being a natural number which we identify with the set of its predecessors, i.e. $c = \{0, 1, \ldots, c-1\}$) is said to be a *colouring function.* It may be interpreted as a colouring of n-element subsets of X by colours $0, 1, 2, \ldots, c-1$. In 1929 the English mathematician F. P. Ramsey proved that if C is a function colouring $[X]^n$ and X is big with respect to c and n then there exists a big set Y such that all its n-element

subsets are coloured by one colour. Such a set $Y \subseteq X$ we call *homogeneous* with respect to C. In fact Ramsey proved the following two theorems:

THEOREM 1 (Infinite Ramsey Theorem). Let n, c be positive natural numbers. For any colouring function $C : [N]^n \to c$ there is an infinite

set $Y \subseteq N$ such that Y is homogeneous with respect to C, i.e. $C \mid [Y]^n$ is constant.

THEOREM 2 (Finite Ramsey Theorem). Let s, n, c be positive natural numbers such that $s \geq n + 1$. Then there is a number $R(s, n, c)$ such that for every $r \geq R(s, n, c)$, for any set X having r elements and any colouring function $C : [X]^n \to c$ there exists a set homogeneous with respect to C having s elements.

These theorems are not intuitively obvious and need proofs. They can be treated as generalizations of Dirichlet's *Schubfachprinzip*. For $n = 1$ Theorem 1 says that if one devides an infinite set into a finite number of disjoint parts then one of these parts must be infinite. Theorem 2 for $n = 1$, $s = 2$ and $R(2, 1, c) = c + 1$ is exactly Dirichlet's principle: if one divides a set containing $c + 1$ elements (or more) into c parts then one of them must contain at least 2 elements.

It turns out that Finite Ramsey Theorem can be proved in PA. Harrington observed that modifying it a bit we obtain a sentence independent of PA. Call a set $X \subseteq N$ *relatively large* iff *card*(X) ε $\min(X)$. Then for example the set $\{2, 3, 89, 92\}$ is relatively large but the set $\{10, 13, 7, 9\}$ is not relatively large. The Paris-Harrington sentence φ_0 says now:

for any natural numbers s, n, c there exists a natural number $H(s, n, c)$ such that for any $h \geq H(s, n, c)$, any set X of cardinality h, any $C : [X]^n \to c$ there is a set Y homogeneous with respect to the function C and such that *card*$Y \geq s$ and Y is relatively large.

It can be proved (for example by set-theoretical methods) that $\mathfrak{N}_0 \models \varphi_0$ (in fact φ_0 is a consequence of Infinite Ramsey Theorem) but PA non-$\models \varphi_0$. Hence φ_0 is an undecidable sentence of a combinatorial contents.

Paris-Harrington sentence is still not a fully satisfying solution to the problem stated above: it has not a purely arithmetical (number-theoretical) contents. Such a sentence was constructed in 1982 by L. Kirby and J. Paris (cf. Kirby and Paris, 1982). The construction uses some ideas of Goodstein (1944). To describe this sentence let m, n be natural numbers and define a *representation of m by the basis n*: we write m as a sum of powers of n (e.g. if $m = 266, n = 2$ then $266 = 2^8 + 2^3 + +2^1$). We do this same with all exponents and at the end we get:

$$266 = 2^{2^{2+1}} + 2^{2+1} + 2^1.$$

We define now a number $G_n(m)$ as follows:

if $m = 0$ then $G_n(m) = 0$,
if $m \neq 0$ then $G_n(m)$ is a number obtained by substituting everywhere in the representation of m (by the basis n) the number n by $n + 1$ and subtracting 1.

For example: $G_2(266) = 3^{3^{3+1}} + 3^{3+1} + 2 \approx 10^{38}$.
The Goodstein sequence for m is defined in the following way:

$$\begin{aligned} m_0 &= m, \\ m_1 &= G_2(m_0), \\ m_2 &= G_3(m_1), \\ &\vdots \end{aligned}$$

For example:

$$\begin{aligned} m_0 &= 266_0 = 2^{2^{2+1}} + 2^{2+1} + 2, \\ m_1 &= 266_1 = G(m_0) = 3^{3^{3+1}} + 3^{3+1} + 2 \approx 10^{38}, \\ m_2 &= 266_2 = G_3(m_1) = 4^{4^{4+1}} + 4^{4+1} + 1 \approx 10^{616}, \\ m_3 &= 266_3 = G_4(m_2) = 5^{5^{5+1}} + 5^{5+1} \approx 10^{10000} \quad \text{etc.} \end{aligned}$$

Observe that this procedure of constructing the sequence m_k can be described in the language L(PA) of Peano arithmetic. Consider now the following sentence φ_1 of L(PA): $\forall m \exists k(m_k = 0)$. It can be proved that $\mathfrak{N}_0 \models \varphi_1$ but PA non-$\models \varphi_1$. The unprovability of φ_1 has its source, roughly speaking, in the fact that $m_k = 0$ only for very big k, e.g. for $m = 4, m_k = 0$ for $k = 3 \cdot 2^{402653211} - 3 \approx 10^{121000000}$ (observe that the whole number of atoms in the Universe is estimated as 10^{80}).

Proofs of Paris-Harington and Kirby-Paris results are technical and cannot be presented here. They use sophisticated machinary of the model theory of arithmetic and of the indicator theory. For some remarks on the proofs as well as for the discussion of sources of undecidability of the considered sentences cf. (Murawski, 1987).

The results of Paris-Harrington and Kirby-Paris being strengthenings of Gödel's incompleteness theorem demonstrated that the hopes described at the beginning of this section cannot be realized.

4. GENERALIZED HILBERT'S PROGRAM

Incompleteness results of Gödel indicated various obstacles in carring out the validation and justification of classical mathematics on finitistic grounds postulated by Hilbert. The natural consequence of it was the idea of extending the admissible methods and allowing general constructive methods. It seems that P. Bernays was among the first to recognize this need. He wrote:

> It thus became apparent that the "finite Standpunkt" is not the only alternative to classical ways of reasoning and is not necessary implied by the idea of proof theory. An enlarging of the methods of proof theory was therefore suggested: instead of a restriction to finitist methods of reasoning, it was required only that the arguments be of a constructive character, allowing us to deal with more general forms of inference. (Cf. Bernays, 1967, p. 502).

One of the motivations of this shift from the original Hilbert's program can be sought in Gödel's reduction (independently found also by G. Gentzen) of Peano arithmetic PA to the intuitionistic system HA of Heyting's arithmetic – it was shown that PA is consistent relative to HA (cf. Gödel, 1933; Gentzen, 1969).[9] Another reason may be Gentzen's proof of consistency of PA by transfinite induction up to ε_0 (cf. Gentzen, 1936; Gentzen, 1938)[10] which was apparently accepted by Hilbert and Bernays in the second volume of *Grundlagen der Mathematik*.

There arises, of course, a problem: what is meant here by constructivity – this concept is in general much less clear than that of finitism. Nevetherless the broadening of original Hilbert's proof theory postulated by Bernays has become an accepted paradigm. Investigations were carried out in this direction and several results were obtained. Studies following the idea of Gentzen of using a transfinite induction on a certain recursive ordering for some ordinal form a part of them (cf. Schütte, 1977; Takeuti, 1987). Another program of reductionism was elaborated by Feferman, namely the program of predicative reductionism (cf. Feferman, 1964–1968, see also Simpson, 1985a). Gödel has proposed an "extension" of the "finite Standpunkt" by way of primitive recursive functionals of higher type (cf. Gödel, 1958).

A further refinement of the original Hilbert's program was suggested in (Kreisel, 1958) and elaborated in (Kreisel, 1968) where a call for a hierarchy of Hilbert's programs can be found. Beside reduction to finitary and constructive conceptions, Kreisel considers also the non-constructive predicative conception and within constructivity itself he analyzes which specific principles would be needed for various pieces

of reductive work.

It is impossible to report here on particular results obtained along those lines (it would require a lot of technical notions and would change completely the character of this paper). One can find a survey of them in Feferman (1988). Let us note only that the researches have led to two surprising insights: (i) classical analysis can be formally developed in conservative extensions of elementary number theory and (ii) strong impredicative subsystems of analysis can be reduced to constructively meaningful theories, i.e. relative consistency proofs can be given by constructive means for impredicative parts of second order arithmetic.

We want to stress at the end of this section that all the generalizations discussed above are very different from the original Hilbert's program. Hilbert's postulate was the validation and justification of classical mathematics by a reduction to finitistic mathematics. The latter was important here for philosophical and, say, ideological reasons: finitistic objects and reasoning have clear physical meaning and are indispensable for all scientific thought. None of the proposed generalizations can be viewed as finitistic (whatever it means). Hence they have another value and meaning from the methodological and generally philosophical point of view. They are not contributing directly to Hilbert's program but on the other hand they are in our opinion compatible with Hilbert's reductionist philosophy.

5. RELATIVISED HILBERT'S PROGRAM AND REVERSE MATHEMATICS

Another consequence of Gödel's incompleteness results (beside those described above) is so called relativized Hilbert's program. If the entire infinitistic mathematics cannot be reduced to and justified by finitistic mathematics then one can ask for which part of it is that possible? In other words: how much of infinitistic mathematics can be developed within formal systems which are conservative over finitistic mathematics with respect to real sentences? This constitutes the relativized version of the program of Hilbert. In what follows we would like to show how the so called reverse mathematics of Friedman and Simpson contributes to it providing us with a partial realization of Hilbert's original program.

To be able to consider the problem one should specify what is meant by finitistic mathematics and by real sentences. We follow here Tait

(1981) where it is claimed that Hilbert's finitism is captured by the formal system PRA of primitive recursive arithmetic (called also Skolem arithmetic). Its language contains the constant 0, successor function S and a function symbol for each primitive recursive function. Its non-logical axioms are: some trivial axioms concerning the constant terms $0, S0, SS0$ etc. and the successor function, the defining equations of the primitive recursive functions and induction on quantifier-free formulas. The theory PRA is certainly finitistic and "logic-free". On the other hand it is powerful enough to accommodate all elementary reasonings about natural numbers and manipulations of finite strings of symbols. By real sentences we shall understand Π_1^0 sentences, i.e. sentences of the language of Peano arithmetic of the form $\forall x \varphi(x, \ldots)$ where φ contains only atomic formulas, connectives and bounded quantifiers.

It turns out that one can formalize classical mathematics not only in set theory but most of its parts (such as geometry, number theory, analysis,

differential equations, complex analysis etc.) can be formalized in a weaker system called second order arithmetic A_2^- (denoted also sometimes as Z_2). This is a system formalized in a language with two sorts of variables: number variables $x, y, z, \ldots$ and set variables $X, Y, Z, \ldots$ Its non-logical constants are those of Peano arithmetic, i.e. $0, S, +, \cdot$ and the membership relation $\in$. Nonlogical axioms of A_2^- are the following:

(1) axioms of PA without the axiom scheme of induction,
(2) (extensionality) $\forall x(x \in X \equiv x \in Y) \rightarrow (X = Y)$
(3) (induction axiom) $0 \in X \;\&\; \forall x(x \in X \rightarrow Sx \in X) \rightarrow \forall x(x \in X)$
(4) (axiom scheme of comprehension) $\exists X \forall x[x \in X \equiv \varphi(x, \ldots)]$,

where $\varphi(x, \ldots)$ is any formula of the language of A_2^- (possibly with free number- or set-variables) in which X does not occur free. Possible models of A_2^- are structures of the form $(\mathfrak{M}, \mathcal{X})$ where $\mathfrak{M}$ is a model of PA and $\mathcal{X}$ is a family of subsets of M (the universe of $\mathfrak{M}$). Observe that for the standard model $\mathfrak{N}_0$ of PA the structure $(\mathfrak{N}_0, \mathcal{P}(N))$ is a model of A_2^- but this is not the case for a non-standard model $\mathfrak{M}$ of PA, i.e. $(\mathfrak{M}, \mathcal{P}(M))$ is never a model of A_2^- (for standard models of A_2^- see e.g. Apt and Marek (1974) and for non-standard models of A_2^- see Murawski (1976–1977) and Murawski (1984a) where a bibliography can also be found).

Second order arithmetic is a nice system because one avoids here troubles connected with set theory (for example paradoxical consequences of the axiom of choice or philosophical problems connected with the adoption of these or those axioms) and on the other hand it is strong enough to prove many important theorems of classical mathematics. There is only a problem of impredicativity of A_2^-: a formula φ in the comprehension axiom may be *any* formula of the language of A_2^-, so it may be of the form $\forall Y \psi(Y, x, \ldots)$, i.e. defining one particular set it may refer to the family of all sets (recall Poincaré's objections against such definitions mentioned in Section 1). But it turns out that in many cases certain fragments of A_2^- suffice, i.e. only particular special forms of the comprehension axiom are needed.

At the Congress of Mathematicians in Vancouver in 1974 H. Friedman formulated a program of foundations of mathematics called today reverse mathematics (cf. Friedman, 1975). Its aim is to study the role of set existence axioms, i.e. comprehension axioms in ordinary mathematics. The main problem is: Given a specific theorem τ of ordinary mathematics, which set existence axioms are needed in order to prove τ? This research program turned out to be very fruitful and led to many interesting results[11] and... showed that Hilbert's program can be partially realized! (There is a rich literature connected with reverse mathematics. The most comprehensive report on it will be the book Simpson (199?), one can consult also Simpson (1985b), Simpson (1987) and Simpson (1988) where a bibliography can be found.)

The procedure used in the reverse mathematics (it reveals the inspiration for its name) is the following: assume we know that a given theorem τ can be proved in a particular fragment $S(\tau)$ of A_2^-. Is $S(\tau)$ the weakest fragment with this property? To answer this question positively one shows that the principal set existence axiom of $S(\tau)$ is equivalent to τ, the equivalence being provable in some weaker system in which τ itself is not provable.

Some specific systems – fragments of A_2^- – arose in this context; the most important are: RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi_1^1 - CA_0$. We shall describe only the first three of them. To do this we need a hierarchy of formulas of the language of A_2^-. By an arithmetical formula we mean a formula containing no quantifiers bounding set-variables. Bounded formulas are arithmetical formulas with only bounded quantifiers, i.e. quantifiers of the form: $\forall x < y, \exists x < y$. The class of all such formulas

is denoted by Δ_0^0. Formulas of the class Σ_1^0 are formulas of the form $\exists x_1 \ldots \exists x_n \psi(x_1, \ldots x_n, y_1, \ldots, y_k)$ where $\psi \in \Delta_0^0$ and formulas of the class Π_1^0 are negations of Σ_1^0 formulas.

The system RCA_0 is a theory in the language of A_2^- based on the following axioms: (i) PA^- (i.e. axioms of Peano arithmetic PA without the axiom scheme of induction), (ii) scheme of induction for Σ_1^0 formulas, i.e.:

$$\varphi(0) \,\&\, \forall x[\varphi(x) \to \varphi(Sx)] \to \forall x \varphi(x),$$

where φ is a Σ_1^0 formula, (iii) (recursive comprehension axiom)

$$\forall x[\varphi(x) \equiv \psi(x)] \to \exists X \forall x[x \in X \equiv \varphi(x)],$$

where φ is Σ_1^0 and ψ is Π_1^0. [Axiom (iii) explains the name RCA_0 of the theory.] can be shown that $(\mathfrak{N}_0, \mathrm{Rec})$ where Rec is the family of all recursive sets is a model of RCA_0.

The theory WKL_0 consists of RCA_0 plus a further axiom known as weak König's lemma (therefore the name WKL_0) which states that every infinite binary tree has an infinite path (this can be formulated in the language of A_2^- using coding). It is stronger than RCA_0 what follows for example from the fact that $(\mathfrak{N}_0, \mathrm{Rec})$ is not a model of WKL_0 (this is a consequence of Gödel's theorem on essential undecidability of Peano arithmetic).

The theory ACA_0 is PA^- plus induction axiom plus arithmetical comprehension, i.e. comprehension scheme for any arithmetical formula (possibly containing set parameters). This theory is not weaker than

WKL_0 because it proves weak König's lemma. It is in fact stronger than WKL_0 what follows from the fact that there are models of WKL_0 consisting of sets definable in $\mathfrak{N}_0$ by formulas of a given class[12] whereas for any model $(\mathfrak{N}_0, \mathcal{X})$ of ACA_0 the family $\mathcal{X}$ must be closed with respect to arithmetical definability.

The specified subsystems of A_2^- are appropriate for particular parts of classical mathematics. In RCA_0 one can construct reals, define notions of the convergence of a sequence, of a continuous function, of Riemann's integrability etc. and prove positive results of recursive analysis and recursive algebra. For example one can prove in RCA_0 that every field has an algebraic closure, that every ordered field has an extension to a real closed field as well as the intermediate value theorem.

The theory WKL_0 turns out to be a quite strong theory, in particular

one can prove in it the following theorems:

- the Heine-Borel covering theorem: every covering of [0, 1] by a countable sequence of open intervals has a finite subcovering,
- every continuous function on [0, 1] is uniformly continuous,
- every continuous function on [0,1] is bounded,
- every continuous function on [0,1] has a supremum,
- every uniformly continuous function on [0,1], which has a supremum, attains it,
- every continuous function on [0,1] attains a maximum value,
- the Hahn-Banach theorem,
- the Cauchy-Peano theorem on the existence of solutions of ordinary differential equations,
- every countable commutative ring has a prime ideal,
- every countable formally real field can be ordered,
- every countable formally real field has a real closure,
- Gödel's completeness theorem for the predicate calculus.

Even more: if S is one of the above stated theorems then $\mathrm{RCA}_0 + S$ is equivalent to WKL_0.

To indicate the strength of ACA_0 let us mention that the following theorems can be proved in it:

- the Bolzano-Weierstrass theorem (every bounded sequence of real numbers has a convergent subsequence),
- every Cauchy sequence of reals is convergent,
- every bounded sequence of reals has a supremum,
- every bounded increasing sequence of real numbers is convergent,
- the Arzela-Ascoli lemma (any bounded equicontinuous sequence of functions on [0, 1] has a uniformly convergent subsequence),
- every countable vector space has a basis,
- every countable commutative ring has a maximal ideal,
- every countable abelian group has a unique divisible closure.

And again, if S is any of those theorems then $\mathrm{RCA}_0 + S$ is equivalent to ACA_0.

Those results indicate how much of A_2^- do we need in fact to prove various particular theorems of classical mathematics. It also appears that proofs (in the formalized subsystems of A_2^-) of uniqueness are usually more difficult and more complicated than proofs of the existence (in mathematical practice the former are usually simple consequences of the latter). There is also no direct connection between the complexity

of a classical proof of a theorem and the level in the hierarchy of subsystems of A_2^- in which a formalized version of it can be proved (as an example can serve here the theorem that every abelian group has a torsion subgroup which is trivial in classical algebra but, as stated above, RCA_0 + this theorem is equivalent to ACA_0, hence it is not a theorem of, say, WKL_0). Results of reverse mathematics have also interesting mathematical, not only logical, applications. For example: since Cauchy-Peano theorem on the existence of solutions of ordinary differential equations is equivalent to WKL_0 and ($\mathfrak{N}_0$, Rec) is not a model of this theory, it follows that there exists a differential equation with a recursive continuous function on the right hand side which has no recursive solution.

Observe that not every mathematical theorem can be classified in Friedman's hierarchy of subsystems of A_2^-. Hilbert's basis theorem can serve here as an example (cf. Simpson, 1988b) – it is provable in ACA_0 but RCA_0 + "Hilbert's basis theorem" is not equivalent to any of the considered systems. There are also sentences unprovable in A_2^- but provable in Zermelo-Fraenkel set theory (cf. Friedman, 1981).

The impatient reader is certainly asking already: well, but what are the connections of reverse mathematics with Hilbert's program? We are just coming to that problem.

In early eighties L. Harrington and Z. Ratajczyk proved a theorem on conservativeness of WKL_0 (none of them published it, the proof can be found in Simpson (199?). It says that if $(\mathfrak{M}, \mathcal{X})$ is a countable model of RCA_0 and $A \in \mathcal{X}$ then there exists a family $\mathcal{Y} \subseteq \mathcal{P}(M)$ such that $A \in \mathcal{Y}$ and $(\mathfrak{M}, \mathcal{Y})$ is a model of WKL_0. To indicate the syntactical context of the theorem recall that Π^1_1 formulas are formulas of the language of A_2^- of the form $\forall X \psi$ where ψ is an arithmetical formula. From Harrington-Ratajczyk theorem follows that the theory WKL_0 is conservative over RCA_0 with respect to Π^1_1 sentences, i.e. that every Π^1_1 sentence provable in WKL_0 can be proved in RCA_0. Friedman showed by modeltheoretical methods that WKL_0 is a conservative extension of PRA with respect to Π^0_2 sentences (cf. Friedman, 1977), this result can be also found in Kirby, Paris (1977). W. Sieg improved this theorem giving an alternative proof (which uses Gentzen-style method) and exhibiting a primitive recursive proof transformation. Thus the reducibility of WKL_0 to PRA is itself provable in PRA.

Combining those results together with the fact that WKL_0 is a strong

theory (as indicated above) we come to the conclusion that a large and significant part of classical mathematics is finitistically reducible. This means in fact a partial realization of Hilbert's program!

All those facts have various "practical" consequences. First observe that the class of Π^0_2 sentences is rather broad – many theorems of number theory can be formulated as sentences belonging to that class. Since one can formalize within WKL_0 the technique of contour integration, any Π^0_2 number-theoretic theorem which is provable with the help of it can also be proved "elementarily", i.e. within PRA and, even more, one can effectively (at least theoretically) find

such an "elementary" proof. To give one more example consider Artin's theorem (being a solution to Hilbert's seventeenth problem – cf. Hilbert (1901)). It can be written as a Π^0_2 sentence. Since all results of the theory of real closed fields needed in the proof of Artin's theorem are provable in WKL_0, it follows by Friedman's and Sieg's theorems that Artin's theorem can be proved in PRA, i.e. in an elementary way.

6. CONCLUSIONS

In this way we came to the end of our story. Conclusions of it are in our opinion rather optimistic. Though the original Hilbert's program of validation and justification of classical mathematics cannot be realized it proved to be fruitful, both philosophically and mathematically by stimulating various investigations which contributed to the clarification of the nature of mathematics. We learnt that not all classical mathematics can be reduced to finitary one, that for a (rather broad) part of it this is still possible, we learnt also the limitations of the axiomatic method. Maybe Hilbert was right claming that "There is no *ignorabimus* in mathematics" (cf. Hilbert, 1926) and finishing his speech over the local radio station in Königsberg (in September 1930, just after the conference during which Gödel announced his incompleteness result) with the motto:

"Wir müssen wissen. Wir werden wissen."
(We must know. We shall know.)

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NOTES

[1] The first paradox was communicated by G. Cantor in a letter to R. Dedekind from 1896, the second in a letter to R. Dedekind from 31st August 1899 (cf. Cantor, 1932, p. 448; see also Murawski, 1984b).

[2] Cf. G. Cantor's letters to R. Dedekind from 28th July 1899 (Cantor, 1932, pp. 443–447) and from 31st August 1899 (Cantor, 1932, p. 448; see also Murawski, 1984b).

[3] Detlefsen writes that "Hilbert did want to preserve classical mathematics, but this was not for him an end in itself. What he valued in classical mathematics was its efficiency (including its psychological naturalness) as a means of locating the truths of real or finitary mathematics. Hence, any alternative to classical mathematics having the same benefits of efficiency would presumably have been equally welcome to Hilbert (cf. Detlefsen, 1990, p. 374).

[4] Gödel explained it by the fact that in the meantime the theorem became well known, hence no such proof was needed any longer.

[5] Let us remark here that Detlefsen's criticism of Smoryński's thesis that Gödel's first incompleteness theorem refutes Hilbert's program is mistaken. The argument of Smoryński does not need in fact the assumption that an ideal theory T is complete with respect to *all* real sentences. It is enough to know that a particular sentence (i.e. Gödel's undecidable sentence) is true and this can be shown for example by set-theoretical methods. Hence we get a sentence which can be proved in an ideal theory but which is undecidable (hence cannot be proved nor disproved) in a real, finitistic part of mathematics.

[6] The first incompleteness theorem requires only that the set of theorems be representable, i.e. the formal system considered may be "identified" with its set of theorems. Consequently, all we need to know about it is its set of theorems. Recall that the second incompleteness theorem requires certain internal properties of the provability to be formally derivable.

[7] Detlefsen's argument is based on the fact that for example $\mathrm{PA} \vdash \mathrm{Con}^{\mathrm{R}}_{\mathrm{PA}}$, where PA denotes Peano arithmetic and $\mathrm{Con}^{\mathrm{R}}_{\mathrm{PA}}$ is the Rosser translation of the metamathematical sentence "PA is consistent" (using Rosser's provability notion $\vdash_{\mathrm{R}}$). So we have here two different notions of provability: Rosser's one $\vdash_{\mathrm{R}}$ and the usual one $\vdash$ based on the classical predicate calculus. To be consequent one should use also in metamathematics the Rosser's provability. But then it should be proved that $\mathrm{PA} \vdash_{\mathrm{R}} \mathrm{Con}^{\mathrm{R}}_{\mathrm{PA}}$.

[8] Theorem XI states that if P is consistent then its consistency is not provable in P, P being Peano arithmetic extended by simple type theory; M is set theory; A is classical analysis.

[9] Gentzen's paper "Über das Verhältniss zwischen intuitionistische und klassische Arithmetik" was submitted and accepted by *Mathematische Annalen* in 1933 but withdrawn on account of Gödel's publication. An English translation was published in 1969 – cf. (Gentzen, 1969).

[10] The first version of Gentzen's proof was submitted in 1935 but was withdrawn after criticism directed against the means used in the proof which were considered to be too strong. This version became publically known because of a paper (Bernays, 1970) and was recently published in the name of Gentzen (cf. Gentzen, 1974).

[11] It is even claimed that the implications of the results of reverse mathematics "make much of what was written in the past on the philosophy of mathematics, obsolete" (cf. Drake, 1987).

[12] A family of Δ^0_2 sets is an example of a model of WKL_0, where Δ^0_2 sets are sets definable in $\mathfrak{N}_0$ simultaneously by Σ^0_2 and Π^0_2 formulas; recall that Σ^0_2 formulas are formulas of the form $\exists x_1 \ldots x_n \psi(x_1, \ldots, x_n, y_1, \ldots, y_k)$, $\psi \in \Pi^0_1$ and Π^0_2 formulas are negations of Σ^0_2 formulas.

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THE LOGIC OF OBJECTS[1]

Traditional assertoric formal logic – in contrast to Aristotelian syllogistic – usually introduces into its language, apart from so-called positive terms, also negative ones. So it employed the functor of nominal negation, 'non', defined with respect to the partial ordering relation described by the functor 'every . . . is . . . '. Inasmuch as the Aristotelian syllogistic (as an inferential system) can be successfully reconstructed as a whole as a theory of partial orderings, as demonstrated by A. Mostowski (1948),[2] traditional formal logic with negative terms cannot be, with respect to the same ordering, extended into a theory of Boolean algebras without running counter to colloquial language and the philosophical tradition. If the relationship between objects described by the functor 'every . . . is . . . ' had a concept of 'object' as its greatest element – for according to the theory of objects 'everything is an object' – then the complement of the last element 'a non-object' would be the first element. Such a 'non-object', being an object (being 'everything' at the same time), would be a contradictory being. With reference to the classical distinction between categorial and transcendental concepts we shall call the consistent beings (as distinguished from the concept of all the objects) – the categorial objects. As a consequence of this we shall be defending two ideas:

1. that treating the word 'is' as a synonym of the expression 'every . . . is . . . ' has a long tradition and is solidly grounded in colloquial language; and
2. that traditional logic is a theory of not one but of two orderings: of the partial ordering between objects described by the word 'is', and of an other partial ordering, between concepts, defined by the expression 'the concept . . . is included in the concept . . . '.

The logical connections between both logical orderings, revealing the sense of the words 'is' and 'is included', will be explored in three logical systems, stemming from traditional logic. We shall call these systems (1) the logic of categorial objects, (2) the logic of concepts and

Jan Woleński (ed.), Philosophical Logic in Poland, 129–152.

3) the logic of systems. These three logical systems are interconnected (especially through their semantics) in such a way that every subsequent system is, in a way, an extension of the previous theory. Together they function as one deductive whole, called here the logic of objects.

1. THE LOGIC OF CATEGORIAL OBJECTS

1.1. We will apply an elementary language with only one sort of variable: $x, y, z, \ldots$ (of the category of names) and with only one primary predicate, 'is'. Let the propositional formula of the sort 'x is y' be presented by the notation '$y(x)$'. This is an elliptical form, in which the word 'is' is being systematically omitted as an understood fragment of the formula. The logical connectives: $\neg$ (negation), $\Rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction) and $\Leftrightarrow$ (equivalence) together with the quantifiers: $\forall$ (universal), $\exists$ (existential) and $\exists_1$ (individual) exhaust the list of all the primary terms of this language.

All the objects are the domain of variability of individual variables in the language. The definition of an 'object' we take from (Łukasiewicz, 1910): 'by an object we ought to mean only something that cannot both have and not have the same feature at the same time [...]. According to this definition we call an "object" everything that is not contradictory'.[3] All that matters is that the objects are consistent. The distinction between individual and general objects has no essential role to play in the logic of objects. The 'emptiness' which can be understood here only as a 'contradiction' is not taken into consideration in the area where we are dealing only with consistent objects.

1.2. Building our theory called the logic of objects we shall apply the Borkowski-Słupecki assumption method of proof, adding our system to quantification logic. In our proofs we shall use the sign '$\vdash$' to signify deductive inference and the sign '::' for inferential equivalence. Primary and secondary rules, on which the inference and inferential equivalence mentioned above are based, will be treated everywhere as well known and easily understood. The proofs starting off with a thesis being already part of the system will be the ordinary, not assumption method proofs – whereas the assumption method proofs start off with any sentential formula which need not be a theorem. An additional assumption sentence will be preceded by Roman numbers, and a parethetical part of the proof (in Borkowski-Słupecki's sys-

tems with the additional numeration), starting off with an extra assumption I, II, ..., will be always closed with the same number, e.g.: ... II $y(x) \vdash \exists z y(z)$ II $\vdash y(x) \Leftrightarrow \exists z y(z) \ldots$

1.2.1. This semantic fact that categorial sentences of the sort 'every x is y' have the same sense as sentences 'x is y' was already defended by Leibniz in his work *De arte combinatoria* of 1666. In this dissertation Leibniz states that: 'der Satz "Sokrates ist der Sohn des Sophroniscus" wird [...] zum Inhalt haben: "Wer immer Sokrates ist, ist der Sohn des Sophroniscus". Man wird auch zutreffend sagen: "Jeder Sokrates ist der Sohn des Sophroniscus", obwohl er ein einziger ist'.[4] This assumption of Leibniz that the sentential form 'x is y' means the same as the expression 'everything that is x is also y' we will be taking axiomatically as a starting point for the logic of objects:

A1. $y(x) \Leftrightarrow \forall z[x(z) \Rightarrow y(z)]$.

From this axiom two theses follow directly: one about the transitiveness and the other about the reflexiveness of the relation described by the word 'is':

L1. $x(x)$.

L2. $y(z) \wedge z(x) \Rightarrow y(x)$.

On the basis of the primary term 'is' we define the so-called Aristotelian operators 'e' (no ... is ...), 'i' (some ... is ...), 'o' (some ... is not ...):

df.e: $xey \Leftrightarrow \forall z[x(z) \Rightarrow \neg y(z)]$,

df.i: $xiy \Leftrightarrow \exists z[x(z) \wedge y(z)]$,

df.o: $xoy \Leftrightarrow \exists z[x(z) \wedge \neg y(z)]$.

From here we also infer a few theses of syllogistic:

L3. xix, by L1 $\vdash x(x) \wedge x(x) \vdash \exists z[x(z) \wedge x(z)]$, df.$i$ $\vdash xix$.

L4. $y(z) \wedge zix \Rightarrow xiy$, by $y(z)$, zix, df.i $\vdash \exists u[z(u) \wedge x(u)]$, I $z(u) \wedge x(u) \vdash z(u), x(u)$, L2 $\vdash y(u) \vdash x(u) \wedge y(u) \vdash \exists u[x(u) \wedge y(u)]$, df.$i$ $\vdash xiy$ I $\vdash \forall u[z(u) \wedge x(u) \Rightarrow xiy] \vdash \exists u[z(u) \wedge x(u)] \Rightarrow xiy \vdash xiy$.

L5. $xoy \Leftrightarrow \neg y(x)$, by df.$o$, A1.

L6. $xey \Leftrightarrow \neg xiy$, by df.$e$, df.$i$.

L7. $y(x) \Rightarrow xiy$, by L1 and df.i.

The formulae L1–L6 here are the axioms of the system of Aristotelian syllogistic formalized by J. Łukasiewicz (1929, 1939, 1951). Hence all the laws of syllogistic (square of opposition, the laws of conversion, main syllogistic modes) are theses of our system.

1.2.2. Let us call the objects x and y identical ($x = y$), when they have the same features:

df.= : $x = y \Leftrightarrow \forall z[z(x) \Leftrightarrow z(y)]$.

In the Borkowski-Słupecki assumption system we obtain the logical theory of identity built upon the narrower functional calculus from one axiom: $x = x$, applying additionally the rule of extensionality for identity.[5] In the logic of objects the axiom mentioned above is a consequence of the definition.

L8. $x = x$, since (from the empty set of assumptions it follows that) $z(x) \Leftrightarrow z(x) \vdash \forall z[z(x) \Leftrightarrow z(x)]$, so by df. =, $\vdash x = x$.

L9. $x = y \Rightarrow y = x$, since $x = y$, by df.= $\vdash \forall z[z(x) \Leftrightarrow z(y)] \vdash y = x$.

L10. $x = y \wedge y = z \Rightarrow x = z$, by df.=.

L11. $y(x) \Leftrightarrow \exists z[z = y \wedge z(x)]$, since I $y(x)$, L8 $\vdash y = y \wedge y(x) \vdash \exists z[z = y \wedge z(x)]$ I, II $\exists z[z = y \wedge z(x)]$, III $z = y \wedge z(x) \vdash z = y, z(x)$, df.= $\vdash \forall u[u(z) \Leftrightarrow u(y)] \vdash y(z) \Leftrightarrow y(y)$, L1 $\vdash y(z)$, L2 $\vdash y(x)$ III $\vdash \forall z[z = y \wedge z(x) \Rightarrow y(x)] \vdash \exists z[z = y \wedge z(x)] \Rightarrow y(x) \vdash y(x)$ II.

L12. $y(x) \Leftrightarrow \exists z[z = x \wedge y(z)]$ (proof analogous to that of L11).

L13. $y(x) \Leftrightarrow \forall z[z = x \Rightarrow y(z)]$, since I $y(x), z = x$, df.= $\vdash \forall u[u(z) \Leftrightarrow u(x)] \vdash x(z) \Leftrightarrow x(x)$, L1 $\vdash x(z)$, L2 $\vdash y(z)$ I, II $\forall z[z = x \Rightarrow y(z)] \vdash x = x \vdash y(x)$, L8 $\vdash y(x)$ II.

L14. $y(x) \Leftrightarrow \forall z[z = y \Rightarrow z(x)]$ (proof analogous to that of L13).

L15. $y(x) \Leftrightarrow \forall z[z(y) \Rightarrow z(x)]$, by L2 and L1.

L16. $\forall z[z(x) \Leftrightarrow z(y)] \Leftrightarrow \forall z[x(z) \Leftrightarrow y(z)]$, since (1) $\forall z[z(x) \Leftrightarrow z(y)] \vdash x(x) \Leftrightarrow x(y), y(x) \Leftrightarrow y(y)$, L1 $\vdash x(y), y(x)$, A1 $\vdash \forall z[x(z) \Leftrightarrow y(z)]$; (2) $\forall z[x(z) \Leftrightarrow y(z)]$, L1 $\vdash x(y), y(x)$, I $z(x)$, L2 $\vdash z(y)$ I $\vdash \forall z[z(x) \Rightarrow z(y)]$ II $z(y)$, L2 $\vdash z(x)$ II $\vdash \forall z[z(y) \Rightarrow z(x)] \vdash \forall z[z(x) \Leftrightarrow z(y)]$.

L17. $x = y \Leftrightarrow \forall z[x(z) \Leftrightarrow y(z)]$, by df.=, L16.

L18. $x = y \Leftrightarrow y(x) \wedge x(y)$, by L17 and A1.

Statements L1 and L2 together with the thesis L18 determine in fact that the relationship between objects described by the word 'is' is a partial ordering.

L19. $y(x) \Leftrightarrow [y(x) \vee x = y]$, since $y(x) :: y(x) \vee [y(x) \wedge x(y)]$, L18.

1.2.3. In higher-order functional calculi (or in the full system of the simple theory of types) the so-called defining axiom is obligatory (or rather a scheme of such axioms), which (in its simpler form) asserts for every formula $A[x]$ (in which there is no free variable F) that $\exists F \forall x(Fx \Leftrightarrow A[x])$. In the logic of categorial objects a postulate determining that for every formula $A[x]$ (with the free variable x, but without the free variable y) of this logic is $\exists y \forall x[y(x) \Leftrightarrow A[x]]$ would be an analogue of the scheme mentioned above. Such a postulate would determine that for every sentential function there would be an object containing these and only these objects that satisfy this function, which is obviously false. For example, there is no object y, such that $\forall x[y(x) \Leftrightarrow \neg x = x]$. The reasoning that has just been given suggests the conclusion that the 'objectivity' (or consistency) of every concept, which could according to the definition be introduced into the logic of objects, should be proved every time. There is actually such an opportunity when we want to add the concept of the complementation of an object to the system that we are building. In order to attain it we accept axiomatically that:

A2. $\forall y \exists z \forall x[z(x) \Leftrightarrow xey]$.

L20. $\forall y \exists_1 z \forall x[z(x) \Leftrightarrow xey]$, by A2 and $\forall x[z(x) \Leftrightarrow xey]$, $\forall x[u(x) \Leftrightarrow xey] \vdash z(z) \Leftrightarrow zey, u(u) \Leftrightarrow uey, z(u) \Leftrightarrow uey, u(z) \Leftrightarrow zey$, L1 $\vdash z(u), u(z)$, L18 $\vdash u = z$.

On the basis of statement L20 we define the object y' ('non-y'), i.e. the object's complement:

df.': $y'(x) \Leftrightarrow xey.$

L21. xex', by L1 $\vdash x'(x')$, df.' $\vdash x'ex \vdash xex'$.

L22. $\neg x'(x)$, by L21 $\vdash xex' \vdash xox'$, L5 $\vdash \neg x'(x)$.

L23. $\neg x(x')$, by L21 $\vdash xex' \vdash x'ox$, L5 $\vdash \neg x(x')$.

L24. $x''(x)$, by df.' $\vdash x''(x) \Leftrightarrow xex'$, L21 $\vdash x''(x)$.

We accept axiomatically the opposite connection:

A3. $x(x'')$.

L25. $x'' = x$, by L24, A3, L18.

The formulae that are the theses L5, L6, L3, L4, L25 of the logic of objects are the axioms of Otto Bird's (1964) formalized system for the positive part of traditional assertoric formal logic (Aristotelian syllogistic together with the square of opposition, laws of conversion, obversion and contraposition).[6] Hence the above mentioned traditional logic is contained in the logic of objects.

Here are some selected theses of the logic of categorial objects supplemented by the concept of complement:

L26. $xiy \Leftrightarrow \neg y'(x)$, by df.', $xiy :: \neg xey :: \neg y'(x)$.

L27. $y(x) \Leftrightarrow xey'$, by L25, df.', $y(x) :: y''(x) :: xey'$.

L28. $xiy \Leftrightarrow xoy'$, by L26, L5.

L29. $xoy \Leftrightarrow xiy'$, by L27, L6.

L30. $\neg y(x) \Leftrightarrow \exists z[x(z) \wedge zey]$, by L5, L29, df.i, df.'.

L31. $xoy \Leftrightarrow \exists z[x(z) \wedge zey]$, by L30, L5.

L32. $y(x) \Leftrightarrow x'(y')$, by L27, df.', $y(x) :: xey' :: y'ex :: x'(y')$.

L33. $xoy \Leftrightarrow y'ox'$, by L32, L5.

L34. $xey \Rightarrow y'ox'$, by L33.

L35. $y(x) \Leftrightarrow \forall z[x(z) \Rightarrow ziy]$, by L27, df.$e$, L26.

L36. $xey \Leftrightarrow \forall z[x(z) \Rightarrow zoy]$, by df.', L35, L29.

L37. $y'(x) \Leftrightarrow \forall z[x(z) \Rightarrow \; zoy]$, by L36, df.'.

L38. $x = y \Leftrightarrow x' = y'$, by L18, L32, L18, $x = y :: x'(y'), y'(x') :: x' = y'$.

L39. $y'(x) \Leftrightarrow x'(y)$, by df.'.

L40. $y(x') \Leftrightarrow x(y')$, by L32, L25.

L41. $y = x' \Leftrightarrow x = y'$, by L18, L40, L39, L18.

L42. $y(x') \Leftrightarrow \forall z[zex \Rightarrow y(z)]$, by A1, df.'.

L43. $y = x' \Leftrightarrow xey \wedge \forall z[zex \Rightarrow y(z)]$, by L18, L39, df.', L42.

The statement L43 says that a complement of the object x is the greatest element among all the objects disjoint from the object x. Although complementation is an operation on categorial objects, it is not a Boolean operation, because the logic of categorial objects excludes the existence both of a logical zero and of a logical one, as follows

L44. $\neg\exists x \forall y y(x)$, by L22 $\vdash \neg x'(x) \vdash \exists y \neg y(x) \vdash \forall x \exists y \neg y(x) \vdash \neg\exists x \forall y y(x)$.

L45. $\neg\exists y \forall x y(x)$, by L23 $\vdash \neg y(y') \vdash \exists x \neg y(x) \vdash \forall y \exists x \neg y(x) \vdash \neg\exists y \forall x y(x)$.

In contrast to the positive part of traditional logic, its negative part contains existential theses. In order to formalize also a negative part of this logic on the basis of a higher-order functional calculus, K. Ajdukiewicz (1926) accepted the axiom:[7]

$$\exists x \exists y \exists z(yez \wedge xez \wedge xey).$$

In the logic of objects we can simplify this axiom to the form:

A4. $\exists x \exists y(xey \wedge x'oy)$, because:

L46. $\exists x \exists y(xey \wedge x'oy) \Leftrightarrow \exists x \exists y \exists z(yez \wedge xez \wedge xey)$, by L29, df.i, L39, df.*e*: $\exists x \exists y(xey \wedge x'oy) :: \exists x \exists y(xey \wedge x'iy') :: \exists x \exists y\{xey \wedge \exists z[x'(z) \wedge y'(z)]\} :: \exists x \exists y \exists z[z'(y) \wedge z'(x) \wedge xey] :: \exists x \exists y \exists z(yez \wedge xez \wedge xey)$.

The logic of objects supplemented by the axiom A4 contains the whole of traditional assertoric formal logic.

1.2.4. A product of two objects, e.g. a square and a circle, is a 'square

circle', which is not an object. So not every product of objects is an object. We accept the following axiom in order to avoid getting out of the sphere of objects through our operations on objects:

A5. $xiy \Rightarrow \exists w \forall z[w(z) \Leftrightarrow x(z) \wedge y(z)]$.

L47. $xiy \Rightarrow \exists_1 w \forall z[w(z) \Leftrightarrow x(z) \wedge y(z)]$, by A5, L1, L18.

The product of objects x and y we write symbolically as '$x \cdot y$', which reads 'xy', hence e.g. in the case of the product a 'lonely · white · sail' – a 'lonely white
sail'.

df.·: $xiy \Rightarrow \forall z[x \cdot y(z) \Leftrightarrow x(z) \wedge y(z)]$.

Here are some selected theses concerning the product of objects:

L48. $\forall z[x(z) \wedge y(z) \Rightarrow x \cdot y(z)]$, by df.$i$, df.·.

L49. $xiy \Rightarrow x(x \cdot y)$, by xiy, $\forall z[x(z) \wedge y(z) \Rightarrow x(z)]$, df.· $\vdash$ $\forall z[x \cdot y(z) \Rightarrow x(z)]$, A1 $\vdash x(x \cdot y)$.

L50. $xiy \Rightarrow y(x \cdot y)$ (the proof is analogous to that of L49).

L51. $xiy \Rightarrow \forall w\{w(x \cdot y) \Leftrightarrow \forall z[x(z) \wedge y(z) \Rightarrow w(z)]\}$, by A1 and df.·.

L52. $xiy \Rightarrow (x \cdot y = y \cdot x)$, by xiy, L51 $\vdash$ I $w(x \cdot y) :: \forall z[x(z) \wedge y(z) \Rightarrow w(z)] :: \forall z[y(z) \wedge x(z) \Rightarrow w(z)] :: w(y \cdot x)$ I $\vdash$ $\forall w[w(x \cdot y) \Leftrightarrow w(y \cdot x)]$, df.= $\vdash x \cdot y = y \cdot x$.

L53. $x \cdot x = x$, by L3, L51, A1, df.=.

L54. $xiy \Rightarrow [y(x) \Leftrightarrow x \cdot y = x]$, by L18, L49, df.·, L1.

L55. $\exists u[x(u) \wedge y(u) \wedge z(u)] \Rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z$, by df.$i$, df.·, $x(u), y(u), z(u) \vdash yiz, xiy \vdash y \cdot z(u), x \cdot y(u) \vdash xiy \cdot z, x \cdot yiz$, L51 $\vdash$ I $w(x \cdot (y \cdot z)) :: \forall u[x \cdot (y \cdot z)(u) \Rightarrow w(u)] :: \forall u[x(u) \wedge y \cdot z(u) \Rightarrow w(u)] :: \forall u[x(u) \wedge y(u) \wedge z(u) \Rightarrow w(u)] :: \forall u[x \cdot y(u) \wedge z(u) \Rightarrow w(u)] :: \forall u[(x \cdot y) \cdot z(u) \Rightarrow w(u)] :: w((x \cdot y) \cdot z) I \vdash \forall w[w(x \cdot (y \cdot z)) \Leftrightarrow w((x \cdot y) \cdot z)]$, df.= $\vdash x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

L56. $xiy \wedge yiw \wedge y(x) \wedge w(z) \Rightarrow y \cdot w(x \cdot z)$, by L49, L2, L48.

L57. $xiy \Rightarrow \forall z\{z = x{\cdot}y \Leftrightarrow x(z) \wedge y(z) \wedge \forall u[x(u) \wedge y(u) \Rightarrow z(u)]\}$, by L18, df.·, L51.

According to the statement L57, a product of objects (if it exists, i.e. if it is an object) is the greatest lower bound (*infimum*).

1.2.5. A union of objects x and y we write symbolically as '$x + y$', which – in accordance to the rules of colloquial language – we read in the simple sentence in relation to the subject: 'x and y', and in relation to the predicate: 'x or y'. Thus for example the expression 'vertebrate (mammal + amphibian + reptile + bird + fish)' reads: 'mammals and amphibians reptiles birds and fishes are the vertebrates', but the expression 'mammal + amphibian + reptile + bird + fish (vertebrate)' reads: 'a vertebrate is a mammal or an amphibian or a reptile or a bird or a fish'.

df.+: $x'iy' \Rightarrow x + y = (x' \cdot y')$.

Because the variables stand for categorial objects, and the complements of these objects are themselves also objects , x' and y' are also objects. A product $x' \cdot y'$ (when $x'iy'$) is an object and its complement $(x' \cdot y')'$ is also an object. In this way a union $x + y$ is guaranteed existence and uniqueness in the logic of objects with the concepts of complement and of product.

Here are some selected theses concerning the union of objects:

L58. $xiy \Rightarrow x' + y' = (x \cdot y)'$, by df.+, L25.

L59. $xiy \Rightarrow x \cdot y = (x' + y')'$, by xiy, L58 $\vdash x' + y' = (x \cdot y)'$, L38 $\vdash (x' + y')' = (x \cdot y)''$, L25 $\vdash (x' + y')' = x \cdot y$, L9 $\vdash x \cdot y = (x' + y')'$.

L60. $x'iy' \Rightarrow \forall z\{x + y(z) \Leftrightarrow \forall u[z(u) \Rightarrow (uix \vee uiy]\}$, by df.+, df.', df.$e$, df.·, L6.

L61. $x'iy' \Rightarrow \forall w\{w(x + y) \Leftrightarrow \forall z[\forall u(z(u) \Rightarrow (uix \vee uiy)) \Rightarrow w(z)]\}$, by A1, L60.

L62. $x'iy' \Rightarrow \forall w\{w = x + y \Leftrightarrow \forall u[w(u) \Rightarrow (uix \vee uiy)] \wedge \forall z[\forall u(z(u) \Rightarrow (uix \vee uiy)) \Rightarrow w(z)]\}$, by L18, L60, L61.

L63. $x'iy' \Rightarrow x + y(x)$, by $x'iy'$, L1 $\vdash x(x)$, L35 $\vdash \forall z[x(z) \Rightarrow zix] \vdash \forall z[x(z) \Rightarrow (zix \vee ziy)]$, L60 $\vdash x + y(x)$.

L64 $x'iy' \Rightarrow x + y = y + x$, by L52, L38, df.+.

L65. $x'iy' \Rightarrow x + y(y)$, by L63, L64.

L66. $x'iy' \Rightarrow [y(x) \Leftrightarrow x + y = y]$, by L54, L32, L38, L25, L64.

L67. $\exists u[x'(u) \wedge y'(u) \wedge z'(u)] \Rightarrow x + (y + z) = (x + y) + z$, by $\exists u[x'(u) \wedge y'(u) \wedge z'(u)]$, L55 $\vdash x' \cdot (y' \cdot z') = (x' \cdot y') \cdot z'$, L38, L48, df.$i$ $\vdash (x' \cdot (y' \cdot z'))' = ((x' \cdot y') \cdot z')', x'iy' \cdot z', x' \cdot y'iz'$, L58, L25 $\vdash x + (y' \cdot z')' = (x' \cdot z')' + z$, df.$i$, df.+, L25 $\vdash x + (y + z) = (x + y) + z$.

L68. $x'iy' \Rightarrow \forall w[w(x + y) \Leftrightarrow w(x) \wedge w(y)]$, by L32, L59, L25, df.·, L32.

L69. $x'iz' \wedge y'iw' \wedge y(x) \wedge w(z) \Rightarrow y + w(x + z)$, by L32, L56, L32, df.+.

L70. $x'iy' \Rightarrow \forall z[x + y(z) \wedge x'(z) \Rightarrow y(z)]$, by $x'iy', x + y(z), x'(z) \vdash$ I $z(u)$, L2 $\vdash x + y(u), x'(u)$, L60, df.' $\vdash uix \vee uiy, uex \vdash \neg uix \vdash uiy$ I $\vdash \forall u[z(u) \Rightarrow uiy]$, L35 $\vdash y(z)$.

L71. $x'iy' \wedge x'iz' \wedge (x+y)i(x+z) \wedge yiz \wedge x'i(y \cdot z)' \Rightarrow (x+y) \cdot (x+z) = x+(y \cdot z)$, by $x'iy', x'iz', (x+y)i(x+z), yiz, x'i(y \cdot z)' \vdash$
(1) L63 $\vdash x + y(x), x + z(x)$, L48 $\vdash (x + y) \cdot (x + z)(x)$, L63, L49, L50 $\vdash x + y(y), y(y \cdot z), x + z(z), z(y \cdot z)$ L2 $\vdash x + y(y \cdot z), x + z(y \cdot z)$, L48 $\vdash (x + y) \cdot (x + z)(y \cdot z)$, L68 $\vdash (x + y) \cdot (x + z)(x + (y \cdot z))$,
(2) I $(x+y) \cdot (x+z)(u)$, df.· $\vdash x+y(u), x+z(u)$, II $u(w), \neg wix$, df.e, df.', L2 $\vdash x'(w), x+y(w), x+z(w)$, L70 $\vdash y(w) \wedge z(w)$, L48 $\vdash y \cdot z(w)$, L7 $\vdash wiy \cdot z$ II $\vdash \forall w[u(w) \Rightarrow (wix \vee wiy \cdot z)]$, L60 $\vdash x+(y \cdot z)(u)$ I $\vdash \forall u[(x+y) \cdot (x+z)(u) \Rightarrow x+(y \cdot z)(u)]$, L35 $\vdash x + (y \cdot z)((x + y) \cdot (x + z))$,
(3) L18, (1), (2) $\vdash (x + y) \cdot (x + z) = x + (y \cdot z)$.

L72. $xiy \wedge xiz \wedge (x' \cdot y')i(x' \cdot z') \wedge y'iz' \wedge xiy \cdot z \Rightarrow (x \cdot y) + (x \cdot z) = x \cdot (y + z)$, by L71, L25, L38, df.+, L59.

L73. $x'iy' \Rightarrow \forall z\{x + y = z \Leftrightarrow z(x) \wedge z(y) \wedge \forall u[u(x) \wedge u(y) \Rightarrow u(z)]\}$, by L18, L68, L69, L25, L32, L51 $\vdash x + y = z :: z(x + y) \wedge x + y(z) :: z(x) \wedge z(y) \wedge z'(x' \cdot y') :: z(x) \wedge z(y) \wedge \forall u[x'(u) \wedge y'(u) \Rightarrow z'(u)] :: z(x) \wedge z(y) \wedge \forall u[x'(u') \wedge y'(u') \Rightarrow z'(u')] :: z(x) \wedge z(y) \wedge \forall u[u(x) \wedge u(y) \Rightarrow u(z)]$.

According to the thesis L73 a union of objects (if it exists, i.e. if it is an object) is the least upper bound (*supremum*).

2. THE LOGIC OF CONCEPTS

2.1. Only objects belong to the field of the relation denoted by the predicate 'is', whereas concepts – regardless of what they are in their nature – are mutually bound by the relation of inclusion. We use a symbol "$\leq$" for the functor 'concept ... is included in concept ... '. Just as the word 'is' is a primary term of the logic of categorial objects, the expression 'is included in' is a primary term of the logic of concepts. Because one can conceive of all sorts of beings, not only categorial objects, but also of contradictory and transcendental beings, we introduce a new kind of individual variable: $a, b, c, \ldots$ which will stand for everything in general, i.e. objects and non-objects.

2.2. To the concept of inclusion we add all the postulates of the theory of Boolean algebras, together with the definitions of the first and the last element, supremum, infimum and a complement.

p1. $a \leq a$.

p2. $a \leq b \wedge b \leq c \Rightarrow a \leq c$.

df1. $a \equiv b \Leftrightarrow a \leq b \wedge b \leq a$ (The expression '$a \equiv b$' reads: 'concepts a and b are (extensionally) equal').

p3. $\exists b \forall a a \leq b$.

df2. $b \equiv 1 \Leftrightarrow \forall a a \leq b$ ('1' – the abbreviation for the name 'object').

p4. $\exists a \forall b a \leq b$.

df3. $a \equiv 0 \Leftrightarrow \forall b a \leq b$ ('0' – the abbreviation for the name 'inconsistent being').

p5. $\forall a \forall b \exists c[a \leq c \wedge b \leq c \wedge \forall d(a \leq d \wedge b \leq d \Rightarrow c \leq d)]$.

df4. $c \equiv a \cup b \Leftrightarrow a \leq c \wedge b \leq c \wedge \forall d(a \leq d \wedge b \leq d \Rightarrow c \leq d)$ ('$a \cup b$' reads: 'a or b').

p6. $\forall a \forall b \exists c[c \leq a \wedge c \leq b \wedge \forall d(d \leq a \wedge d \leq b \Rightarrow d \leq c)]$.

df5. $c \equiv a \cap b \Leftrightarrow c \leq a \wedge c \leq b \wedge \forall d(d \leq a \wedge d \leq b \Rightarrow d \leq c)$ ('$a \cap b$' reads: 'ab').

p7. $(a \cup b) \cap c \equiv (a \cap c) \cup (b \cap c)$.

p8. $\forall a \exists b(a \cup b \equiv 1 \wedge a \cap b \equiv 0)$.

df6. $b \equiv -a \Rightarrow a \cup b \equiv 1 \wedge a \cap b \equiv 0$ ('$-a$' reads: 'non-a').

p9. $\neg(1 \leq 0)$.

The terms (meaningful nominal expressions) of the logic of concepts are: the terms of the logic of categorial objects, new variables $a, b, c, \ldots$, the constants '0' and '1', and all sorts of combined terms, constants or variables mentioned above by the signs '$-$', '$\cup$', '$\cap$'.

Whereas the concept of 'object' contains all the concepts, the concept of a non-consistent being, as an empty concept, is included in every concept. These assignments are expressed on the first two theses:

t1. $\forall a a \leq 1$, by p1 $\vdash 1 \leq 1$, df1 $\vdash 1 \equiv 1$, df2 $\vdash \forall a a \leq 1$.

t2. $\forall b 0 \leq b$, by p1 $\vdash 0 \leq 0$, df1 $\vdash 0 \equiv 0$, df3 $\vdash \forall b 0 \leq b$.

All further laws from the theory of Boolean algebras we assume as well known, but we define, however, the (elliptical) word 'is', in its new sense:

df7. $b(a) \Leftrightarrow \neg(a \leq 0) \wedge a \leq b$.

According to the definition, a is b, when the concept a is not contained in the concept of a non-consistent being and at the same time the concept a is contained in the concept b.[8] We now concentrate on demonstrating a series of statements describing logical connections between the relationships *is* and *is included*.

t3. $b(a) \Rightarrow a \leq b$, by df7.

t4. $a \leq 0 \Rightarrow \neg b(a)$, df7.

t5. $\forall a \neg a(0)$, by $a(0)$, p1, df7 $\vdash 0 \leq 0, \neg(0 \leq 0) \vdash$ contradition.

t6. $1(a) \Leftrightarrow \neg(a \leq 0)$, by df7, t1.

t7. $\neg 1(0)$, by t5.

t8. $\neg 0(0)$, by t5.

t9. $b(a) \Leftrightarrow 1(a) \wedge a \leq b$, by df7, t6.

t10. $1(a) \Rightarrow [b(a) \Leftrightarrow a \leq b]$, by t9.

t11. $a(a) \Leftrightarrow 1(a)$, by t9 and p1.

t12. $b(a) \Leftrightarrow a(a) \wedge a \leq b$, by t9 and t11.

t13. $1(a) \wedge a \leq b \Rightarrow 1(b)$, by $1(a), a \leq b$, t6 $\vdash \neg(a \leq 0)$, p2 $\vdash a \leq b \wedge b \leq 0 \Rightarrow a \leq 0 \vdash \neg(b \leq 0)$, t6 $\vdash 1(b)$.

t14. $b(a) \Rightarrow 1(b)$, by t9, t13.

t15. $1(1)$, by t6 and p9.

t16. $a \leq b \wedge a(c) \Rightarrow b(c)$, by t9, p2, t9.

t17. $b(a) \wedge a(c) \Rightarrow b(c)$, by t3, t16.

t18. $1(a) \Rightarrow \{b(a) \Leftrightarrow \forall c[a(c) \Rightarrow b(c)]\}$, by t17 and t11.

t19. $\neg b(a) \Leftrightarrow [\neg 1(a) \vee \neg(a \leq b)]$, by t9.

t20. $-b(a) \Leftrightarrow 1(a) \wedge a \leq -b$, by t9.

t21. $1(a) \Rightarrow [-b(a) \Leftrightarrow a \leq -b]$, by t20.

t22. $\neg(-b(a)) \Leftrightarrow [\neg 1(a) \vee \neg(a \leq -b)]$, by t20.

t23. $b(-a) \Leftrightarrow 1(-a) \wedge -a \leq b$, by t9.

t24. $1(-a) \Rightarrow [b(-a) \Leftrightarrow -a \leq b]$, by t23.

t25. $-b(-a) \Leftrightarrow 1(-a) \wedge b \leq a$, by t23.

t26. $1(-a) \Rightarrow [-b(-a) \Leftrightarrow b \leq a]$, by t25.

Applying the concept of the relationship 'is' (already introduced) we define now the relation of identity in the logic of concepts:

df8. $a = b \Leftrightarrow b(a) \wedge a(b)$.

In the following statements we describe a logical connection between the identity relation among objects and the equality of concepts:

t27. $a = b \Rightarrow a \equiv b$, by df8, t3, df1.

t28. $a = b \Leftrightarrow 1(a) \wedge a \equiv b$, by df8, t9, t13, df1.

t29. $1(a) \Rightarrow (a = b \Leftrightarrow a \equiv b)$, by t28.

t30. $\exists a \exists b[a \equiv b \wedge \neg(a = b)]$, by $0 \equiv -1$, t5 $\vdash \neg(-1(0))$, df8 $\vdash \neg(0 = -1)$.

t31. $1(-a) \Leftrightarrow \neg(1 \leq a)$, by t6 $\vdash 1(-a) :: \neg(-a \leq 0) :: \neg(1 \leq a)$.

t32. $b(-a) \Leftrightarrow \neg(1 \leq a) \wedge -a \leq b$, by t23, t31.

t33. $\neg b(-a) \Leftrightarrow [1 \leq a \vee \neg(-a \leq b)]$, by t32.

t34. $-b(-a) \Leftrightarrow \neg(1 \leq a) \wedge b \leq a$, by t25, t31.

t35. $\neg(-b(-a)) \Leftrightarrow [1 \leq a \vee \neg(b \leq a)]$, by t34.

2.3. While comparing the semantics of the logic of concepts with the semantics of the logic of categorial objects we see that the models for the logic of concepts are not-degenerate Boolean algebras, whereas the models for the logic of categorial objects are relational systems generated from Boolean algebras (containing at least 8 elements, due to the axiom A4). This happens through taking the universe of these algebras as the universe of the generated systems, but the universe is taken without the first and last elements of the algebras, and the ordering relation is cut down to such a limited universe. Following in the reverse order the relation of mutual dependence between both logics we can notice that the models of the logic of concepts are only these Boolean algebras which have the universe of the model of the logic of categorial objects as their last element. The semantic dependence between both logics above is expressed by the only proper axiom of the logic of concepts:

p10. $a \leq 0 \vee 1 \leq a \vee \exists x x = a$.

On the basis of the added postulate we prove the remaining theorems.

t36. $\forall x 1(x)$, by t11 and L1.

t37. $\forall x \neg(x \leq 0)$, by t6 and t36.

t38. $\forall x \neg(x = 0)$, by t37, t2, df1.

t39. $\forall x \neg(1 \leq x)$, by $1 \leq x$, t36 $\vdash 1(z) \vdash \forall z x(z) \vdash \exists x \forall z x(z)$, L45 $\vdash$ contrad.

t40. $\forall x \neg(x = 1)$, by t39, t1, t27, df1.

t41. $\neg \exists x(x = 1 \vee x = 0)$, by t38 and t40.

t42. $\neg(-x \leq 0)$, by t39.

t43. $\neg(1 \leq -x)$, by t37.

t44. $\exists x x = a \Leftrightarrow \neg(a \leq 0) \wedge \neg(1 \leq a)$, by p10, t37, t39.

t45. $1(a) \Leftrightarrow (1 \leq a \vee \exists x x = a)$, by t6, p10, p2, p9, t6, t44, t6.

t46. $\forall x \exists z z = -x$, by t44, t42, t43.

t47. $b(a) \Rightarrow \neg(-b(a))$, by df7.

t48. $a \cap b \equiv 0 \Leftrightarrow a \leq -b$ (a theorem from the theory of Boolean algebras).

t49. $xey \Leftrightarrow -y(x)$, by (1) xey, df.$e \vdash \forall z[x(z) \Rightarrow \neg y(z)], x \cap y \equiv 0 \vee \neg(x \cap y \equiv 0)$, I $\neg(x \cap y \equiv 0)$, df1, t2 $\vdash \neg(x \cap y \leq 0)$, t6 $\vdash 1(x \cap y), x \cap y \leq x, x \cap y \leq y$, t9 $\vdash x(x \cap y), y(x \cap y)$, t39 $\vdash \neg(1 \leq x \cap y)$, p10 $\vdash \exists z z = x \cap y, x(x \cap y) \Rightarrow \neg y(x \cap y) \vdash \neg y(x \cap y) \vdash$ contrad. I $\vdash x \cap y \equiv 0$, t48 $\vdash x \leq -y$, t39, t9 $\vdash -y(x)$; (2) $-y(x)$, I $x(z)$, t17 $\vdash -y(z)$, t47 $\vdash \neg y(z)$ I, df.e $\vdash xey$.

t50. $-x = x'$, by L43, t46 $\vdash -x = x' \Leftrightarrow xe - x \wedge \forall z[zex \Rightarrow -x(z)]$, t49 $\vdash xe - x \Leftrightarrow -(-x(x)) \vdash xe - x$, t49 $\vdash \forall z[zex \Rightarrow -x(z)] \vdash -x = x'$.

t51. $xiy \Rightarrow \neg(x \cap y \leq 0)$, by $xiy, x \cap y \leq 0$, df.$i \vdash \exists z[x(z) \wedge y(z)] \vdash$ I $x(z), y(z)$, t3 $\vdash z \leq x, z \leq y$, df5 $\vdash z \leq x \cap y$, p2 $\vdash z \leq 0$, t4 $\vdash \neg x(z) \vdash$ contrad. I.

t52. $\neg(1 \leq x \cap y)$, by $x \cap y \leq x$, t39, p2.

t53. $xiy \Rightarrow \exists z z = x \cap y$, by t44, t51, t52.

t54. $xiy \Rightarrow x(x \cap y)$, by t9, t45, t52, t53.

t55. $xiy \Rightarrow y(x \cap y)$, by t9, t45, t52, t53.

t56. $a(c) \wedge b(c) \Rightarrow a \cap b(c)$, by t9, df5, t9.

t57. $c(a) \wedge c(b) \Rightarrow c(a \cup b)$, by t9, df4, t13, t9.

t58. $xiy \Rightarrow x \cap y = x \cdot y$, by t53, L57, t54, t55, t56.

t59. $x'iy' \Rightarrow x \cup y = x + y$, by $x'iy'$, t58 $\vdash x' \cap y' = x' \cdot y'$, t50 $\vdash -x \cap -y = x' \cdot y'$, L38 $\vdash (-x \cap -y)' = (x' \cdot y')'$, t46, t53, t50, df.+ $\vdash -(-x \cap -y) = x + y$.

t60. $1 \leq a \Leftrightarrow \forall x a(x)$, by I $1 \leq a$, t36 $\vdash 1(x)$, t3, p2 $\vdash x \leq a$, t9 $\vdash a(x) \vdash \forall x a(x)$ I; II $\forall x a(x) \vdash a(x), a(x')$, t3, t50 $\vdash x \cup -x \leq a \vdash 1 \leq a$ II.

t61. $\neg(1 \leq a) \Leftrightarrow \exists x \neg a(x)$, by t60.

t62. $\forall x - a(x) \Leftrightarrow \forall x \neg a(x)$, by
(1) $\forall x - a(x) \vdash -a(x)$, t47 $\vdash \neg a(x) \vdash \forall x \neg a(x)$;
(2) $\forall x \neg a(x)$, p10 $\vdash$ I $1 \leq a$, t36 $\vdash a(x) \vdash$ contrad. I $\vdash \neg(1 \leq a)$, II $\exists z z = a \vdash \exists z[z = a \wedge \forall x \neg a(x)] \vdash \exists z \forall x \neg z(x)$, L1 $\vdash z(z) \vdash \exists x z(x) \vdash \forall z \exists x z(x) \vdash \neg \exists z \forall x \neg z(x) \vdash$ contrad. II $\vdash \neg \exists z z = a$, p10 $\vdash a \leq 0 \vdash 1 \leq -a \vdash x \leq -a$, t36, t9 $\vdash -a(x) \vdash \forall x - a(x)$.

t63. $a \leq 0 \Leftrightarrow \forall x \neg a(x)$, by t60, t62.

t64. $\neg(a \leq 0) \Leftrightarrow \exists x a(x)$, by t63.

t65. $1(a) \Leftrightarrow \exists x a(x)$, by t64, t6.

t66. $\exists x x = a \Leftrightarrow \exists x a(x) \wedge \exists x \neg a(x)$, by t44, t64, t61.

3. THE LOGIC OF SYSTEMS

3.1. The concepts of concepts we call systems of concepts, and the logic of systems is in fact the logic of systems of concepts. A new domain of variation of variables makes it necessary to introduce new individual variables: $A, B, C, \ldots$ We use the symbol '$\leq$' as the primary predicate of the logic of systems. '$A \leq B$' reads: 'system of concepts A is included in system of concepts B', in contrast to the expression 'the concept a is included in the concept b'. A similar procedure occurs in the theory of Boolean algebras we thus admit the ambiguity of signs, which are disambiguated only in context. We thus mark the last element (the greatest system) by 'U', and the first element (an empty system) – by '$\emptyset$'. We read the remaining symbols in the same way as we do in the logic of concepts: $A \cup B$ (A or B), $A \cap B (AB)$, $-A$ (*non-A*). Postulates of the theory of Boolean algebras we obtain automatically from the postulates p1, ..., p9 and the definitions Def1, ..., Def6 from

the definitions df1, ..., df6 by replacing the variables 'a', 'b', ... by variables 'A', 'B', ..., the constant '1' by the symbol 'U' and '0' by the symbol '$\emptyset$'. Through the same operations we obtain also the first two statements of the theory of systems: Tw1 from t1 and Tw2 from t2.

3.2. Boolean postulates and definitions we supplement by the additional proper axiomatics, in which we read the marks 'a^*' as 'that which is a' and 'a^+' as 'that which a is'. Because a^* and a^+ themselves are already concepts, their annotation in the context of the functor '$\leq$' reads: 'concept of that which is a' and 'concept of that which a is'. We note that terms of the language of the logic of systems are: individual variables $A, B, C, \ldots$ constants 'U' and '$\emptyset$', expressions t^*, t^+, where t is a term of the logic of concepts, and all the combinations of these variables, constants or terms joined together by the functors '$-$', '$\cup$' and '$\cap$'.

P10. $U \equiv 1^* \wedge 1^* \leq 0^+$.

P11. $a^* \leq b^* \Leftrightarrow a \leq b$.

P12. $a^+ \leq b^+ \Leftrightarrow b \leq a$.

P13. $a^* \leq b^+ \Rightarrow b \leq 0$.

P14. $a^+ \leq b^* \Rightarrow 1 \leq b$.

P15. $a^* \cap a^+ \leq b^* \Rightarrow a \leq b$.

P16. $a^* \cap a^+ \leq b^+ \Rightarrow b \leq a$.

From every element of the universe of the model of the logic of concepts we can form – due to the ordering relation contained in this model – the prime ideal and the principal filter. Then the element marked by 'a' is an element generating the prime ideal (marked 'a^*') and the principal filter (marked 'a^+'). Taking the greatest prime ideal 1^* as a starting point, we obtain a model of the logic of systems in a power set of the ideal mentioned above (the relation of inclusion is reduced to it).

On the basis of the logic of systems we define the new sense of the (elliptical) word 'is' in the following way:

Def7. $B(A) \Leftrightarrow \neg(A \leq 0^*) \wedge A \leq B$.

The statements of the logic of systems Tw3–Tw30 as well as the proofs and the definition Def8 are obtained automatically from the statements, definitions and proofs of the logic of concepts by the replacing all small

letters $a, b, c, \ldots$ by capital letters $A, B, C, \ldots$, '1' – by '1*', '0' by '0*', names of postulates pn – by 'Pn', definitions 'df$k$' – by 'Def$k$' and theses 'ti' – by 'Twi' ($n \leq 9, k \leq 8, 3 \leq i \leq 30$).

Tw31. $\forall A \neg A(0^*)$, since Def7, P1. (Tw31 says that that which is a contradictory being is nothing).

Tw32. $\neg 1^*(0^*)$, by P1, Tw6.

Tw33. $a^* \leq b^* \Leftrightarrow b^+ \leq a^+$, by P11 and P12.

The statement Tw33 says that the concept of that which is a is contained in the concept of that which b is, when the concept of that which b is is contained in the concept of that which a is. (Species of the concept a are included in species of the concept b, when the genera of concept b are included in the genera of concept a.)

Tw34. $\forall a \exists A A \equiv a^*$, by $a^* \equiv a^*$.

Tw35. $\forall a \exists A A \equiv a^+$, by $a^+ \equiv a^+$.

Tw36. $a^* \leq b^+ \Leftrightarrow b \leq 0$, by P13 and $b \leq 0$, P12 $\vdash 0^+ \leq b^+$, Tw1 $\vdash a^* \leq 1^*$, P10 $\vdash a^* \leq 0^+$, P2 $\vdash a^* \leq b^+$.

Tw37. $a^+ \leq b^* \Leftrightarrow 1 \leq b$, by P14, P11, Tw1, P2.

Tw38. $1(b) \Leftrightarrow \neg(a^* \leq b^+)$, by t6, Tw36.

Tw39. $1(b) \Leftrightarrow \neg \exists a(a^* \leq b^+)$, by Tw38.

Tw40. $1(-b) \Leftrightarrow \neg(a^+ \leq b^*)$, by t31, Tw37.

Tw41. $1(-b) \Leftrightarrow \neg \exists a(a^+ \leq b^*)$, by Tw40.

Tw42. $a^* \leq b^+ \Leftrightarrow a^+ \leq (-b)^*$, by Tw36, Tw37.

Tw43. $a^+ \leq b^* \Leftrightarrow a^* \leq (-b)^+$, by Tw37, Tw36.

Tw44. $\neg \exists x \exists y x^* \leq y^+$, by t37, Tw36.

Tw45. $\neg \exists x \exists y x^+ \leq y^*$, by Tw37, t39.

Tw46. $\neg \exists x(x^* \leq x^+ \vee x^+ \leq x^*)$, by Tw44, Tw45.

Tw47. $\forall a \neg(a^+ \leq 0^*)$, by Tw37, p9.

Tw48. $\forall a 1^*(a^+)$, by Tw47, Tw6.

Tw49. $B(a^+) \Leftrightarrow a^+ \leq B$, by Def7 and Tw47.

Tw50. $1(a) \Leftrightarrow [b^*(a^*) \Leftrightarrow b^+ \leq a^+]$, by t6, P11, Def7, Tw33, Tw49.

Tw51. $b^+(a^+) \Leftrightarrow b \leq a$, by Tw49 and P12.

Tw52. $1^*(a^*) \leftrightarrow 1(a)$, by Tw6, P11, t6.

Tw53. $b^*(a^*) \Leftrightarrow b(a)$, by Tw9, Tw52, P11, t9.

Tw54. $1(b) \Rightarrow [b^+(a^+) \Leftrightarrow a(b)]$, by Tw51, t9.

Tw55. $a^* = b^* \Leftrightarrow a = b$, by Def8, Tw53.

Tw56. $a^+ = b^+ \Leftrightarrow a \equiv b$, by Def8, Tw51, df1.

3.3. Let us take the expression '$[a]$' (read: 'being of a'), which is – in a model – an equivalent of a one-element family of sets determined by a product of the prime ideal and the principal filter (both with the same generating element, denoted by 'a').

Def9. $[a] \equiv a^* \cap a^+$ (The concept of being a is the product of the concept of that which is a and of what a is).

Tw57. $[a] = [b] \Leftrightarrow a = b$, by I $[a] = [b]$, Def9 $\vdash a^* \cap a^+ = b^* \cap b^+$, Tw28 $\vdash 1^*(a^* \cap a^+), a^* \cap a^+ \leq b^*, b^* \cap b^+ \leq a^*$, P15 $\vdash a \leq b, b \leq a$, Tw13 $\vdash 1^*(a^*), a \equiv b$, Tw28 $\vdash a = b$ I. II $a = b$, Tw55, Tw56 $\vdash a^* = b^*, a^+ = b^+ \vdash a^* \cap a^+ = b^* \cap b^+$, Def9 $\vdash [a] = [b]$ II.

Tw58. $1^*([a]) \Leftrightarrow 1(a)$, by I $1^*([a])$, Def9 $\vdash 1^*(a^* \cap a^+)$, Tw13 $\vdash 1^*(a^*)$, Tw52 $\vdash 1(a)$ I. II $1(a)$, t6 $\vdash \neg(a \leq 0)$, P15 $\vdash a^* \cap a^+ \leq 0^* \Rightarrow a \leq 0 \vdash \neg(a^* \cap a^+ \leq 0^*)$, Tw6, Def9 $\vdash 1^*([a])$ II.

Tw59. $[a] \leq b^* \Leftrightarrow a \leq b$, by P15 and Def9.

Tw60. $b^*([a]) \Leftrightarrow b(a)$, by Tw9, t9, Tw58, Tw59.

Tw61. $[a] \leq b^+ \Leftrightarrow b \leq a$, by P16 and Def9.

Tw62. $b^+([a]) \leftrightarrow a(b)$, by Tw9, t9, Tw48, Tw61.

Tw63. $[1] \equiv 1^+$, by $1^+ \leq 1^* \vdash 1^+ \equiv 1^* \cap 1^+$, Def9 $\vdash 1^+ \equiv [1]$.

Tw64. $[1] = 1^+$, by Tw28, p9, Tw58, Tw63.

Tw65. $[0] \equiv 0^*$, by P10 $\vdash 0^* \leq 0^+ \vdash 0^* \equiv 0^* \cap 0^+$, Def9 $\vdash 0^* \equiv [0]$.

Tw66. $\neg([0] = 0^*)$, by P1 $\vdash 0^* \leq 0^*$, Tw6 $\vdash \neg 1^*(0^*)$, Tw28 $\vdash \neg(0^* = [0])$.

Using the concept of 'being something' we can introduce one other predicate 'is', whose sense is close to that of a set theory 'is an element of':

Def10. $a \in B \Leftrightarrow [a] \leq B$.

Tw67. $a \in b^* \Leftrightarrow a \leq b$, by Tw59 and Def10.

Tw68. $a \in b^+ \leftrightarrow b \leq a$, by Tw61 and Def10.

Tw69. $a \in a^* \wedge a \in a^+$, by Tw67, Tw68, p1.

Tw70. $(a^* \leq B \vee a^+ \leq B) \Rightarrow a \in B$, by $A \cap B \leq A, A \cap B \leq B$, Def9, Def10.

Tw71. $a \in b^* \Leftrightarrow b \in a^+$, by Tw67, Tw68.

Tw72. $1(a) \Rightarrow [a \in b^* \Leftrightarrow b(a)]$, by t10, Tw67.

Tw73. $1(b) \Rightarrow [a \in b^+ \Leftrightarrow a(b)]$, by t10, Tw68.

Tw74. $\exists a \exists b[a \in b^* \wedge \neg b(a)]$, by $0 \leq b$, Tw67 $\vdash 0 \in b^*$, t5 $\vdash \neg b(0)$.

At the end we define a concept of a categorial object:

Def11. $a \in$ *categorial* $\Leftrightarrow \neg(a \in 0^* \vee a \in 1^+)$.

Tw75. $\forall x x \in$ *categorial*, by Def11, Tw67, t37, Tw68, t39.

4. ONTOLOGY IN SYMBOLIC LANGUAGE

The philosophical theory of being or ontology is the discipline semantically closest to the logic of objects. It can use its formal apparatus, especially at the stage of formalization.

The extension of the logic of objects to the formalized version of the theory of being occurs through enriching the language of this logic with a number of ontological terms. The choice of terminology is a matter

of agreement. Let us illustrate the procedure with some examples.

At the beginning we introduce the concept of an individual. Extralogical constants added to the logic of concepts are marked by capital letters (we use abbreviations in the case of longer names) in the natural language. The name 'individual' is shortened to the form '*IND*'.

DF1. $a \leq \text{IND} \Leftrightarrow \forall x \forall z[a(x) \wedge x(z) \Rightarrow z(x)]$.

The concept of an individual includes every individual, hence every concept contained in it possesses existential subjects exclusive to itself. Because we do not consider this concept to be empty, we admit that:

p11. $\neg\,(\mathit{IND} \leq 0)$,

i.e., in accordance with t6, that $1(\mathit{IND})$, or according to t64, that $\exists x$ $\mathit{IND}(x)$.

TW1. $\mathit{IND}(a) \Leftrightarrow 1(a) \wedge a \leq \mathit{IND}$, by t9 and DF1.

TW2. $\mathit{IND}(y) \Leftrightarrow \forall x \forall z[y(x) \wedge x(z) \Rightarrow z(x)]$, by TW1, t36.

TW3. $\mathit{IND}(y) \Leftrightarrow \forall x[y(x) \Rightarrow x(y)]$, by I $\mathit{IND}(y)$, TW2 $\vdash$ $\forall x \forall z[y(x) \wedge x(z) \Rightarrow z(x)] \vdash y(y) \wedge y(x) \Rightarrow x(y)$, L1 $\vdash \forall x[y(x) \Rightarrow x(y)]$ I. II $\forall x[y(x) \Rightarrow x(y)] \vdash y(z) \Rightarrow z(y) \vdash$ III $y(x), x(z) \vdash z(y)$ III $\vdash \forall x \forall z[y(x) \wedge x(z) \Rightarrow z(y)]$, DF1 $\vdash$ $\mathit{IND}(y)$ II.

TW4. $\mathit{IND}(y) \Leftrightarrow \forall x[y(x) \Rightarrow x = y]$, by TW3 and L18.

TW5. $\neg \forall x\ \mathit{IND}(x)$, by $\forall x\ \mathit{IND}(x) \vdash \forall x[x(z) \Rightarrow z(x)]$, A4, L46 $\vdash \exists x \exists y \exists z(yez \wedge xez \wedge xey) \vdash$ I yez, xez, xey, df.' $\vdash z'(x), y'(x)$, df.i $\vdash z'iy'$, L63 $\vdash z + y(z) \vdash z(z + y)$, L68 $\vdash z(y) \vdash yiz, yez \vdash$ contrad. I.

TW6. $\neg(1 \leq \mathit{IND})$, by TW5, t60.

The postulate p11 and the thesis TW6 claim together that the concept of an individual belongs to the categorial objects.

There is no objection to our admission that the logics of concepts and systems include also relative concepts, e.g. Maria's father, the top of a mountain, a commander of a squad etc. Natural language allows us to create such relatives (relative concepts) from a combination of any two concepts. The latter one always has to take the genitive form. A construction of the sort 'y of x' we will symbolize as 'y/x'.

We assume that:

p12. $b/a \leq b$ and P17. $B/a \leq B$, entails from here that

t67. $b(b/a) \Leftrightarrow 1(b/a)$, by t9 and p12.

A relative concept, of significant importance for ontology, is the concept of a part (improper). The expression '*PART*/x' reads: 'a part of x' and we understand it according to the postulates:

p13. *IND*$(x) \Rightarrow$ *PART*/$x(x)$.

p14. *IND*$(x)\wedge$ *IND*$(y)\wedge$ *PART*/$y(x)\wedge$ *PART*/$x(y) \Rightarrow x = y$.

p15. *IND*$(x)\wedge$ *IND*$(y)\wedge$ *IND*$(z)\wedge$ *PART*/$z(x)\wedge$ *PART*/$y(z) \Rightarrow$ *PART*/$y(x)$.

An individual can be either simple or complex.

DF2. *IND*$(x) \Rightarrow \{$*SIMPLE*$(x) \Leftrightarrow \forall z[$*PART*/$x(z) \Rightarrow x = z]\}$.

The central concept of ontology is the concept of being. Treating it as *per se nota* we only assume that it is a categorial concept, i.e.

p16. $\neg$ (*BEING*$\leq 0 \vee 1 \leq$ *BEING*).

TW7. 1 (*BEING*) $\wedge$ 1(-*BEING*), by p16, t6, t31.

TW8. $\exists x$ *BEING*$(x) \wedge \exists x \neg$ *BEING*(x), by TW7, t64, t61.

Some of the more complex ontological concepts can be introduced on the basis of the logic of systems. Extralogical constants added to the language of this logic we write with small letters in the natural language. Let us define – as an example – some concepts of classical ontology.

DF3. *subject*/$a \equiv a^*$ (The concept of an existential subject of A overlaps with the concept of that which is a).

DF4. *feature*/$a \equiv a^+$ (The concept of a feature of a overlaps with the concept of that which a is).

DF5. *BEING*$(x) \Rightarrow$ *substance*/$x = x^*$.

DF6. *BEING*$(x) \Rightarrow$ *accident*/$x = x^+$.

TW9. *BEING*$(x) \Rightarrow$ *substance*/x = *subject*/x, by DF5 and DF3.

TW10. *BEING* $(x) \Rightarrow$ *accident*/x = *feature*/x, by DF6 and DF4.

TW11. *feature*/$a(b^+) \Leftrightarrow a \leq b$, by DF4, Tw51.

TW12. $1(a) \Rightarrow [feature/a(b^{+}) \Leftrightarrow b(a)]$, by DF4, Tw54.

DF7. *existence* = [*BEING*] (Existence is the same as being a being).

DF8. $essence/a = [a]$ (The nature of a is its being a).

From both definitions (DF7 and DF8) follows straight away:

TW13. *essence/BEING = existence.*

The outline of the logic of objects presented above together with an illustration of its ontological applications is only a preliminary proposal as to the senses of the word 'is' that so far has been little explored in its formal-logical consequences and in its coherence with colloquial language.

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NOTES

[1] Translated by Beata Agnieszka Błaszczyk.
[2] (Mostowski, 1948) chapt. VI, part 3.
[3] (Łukasiewicz, 1910) p. 119.
[4] (Leibniz, 1960) p. 38.
[5] (Słupecki et al., 1963) pp. 99–104.
[6] The positive part of traditional assertoric formal logic was axiomatized before, but in a different way by Wedberg, (1948), Thomas (1949), Meredith (1953) and Shepherdson (1956).
[7] (Ajdukiewicz, 1960) p. 21.
[8] As is easily noticed, the sense of the word 'is' in the logic of concepts – in relation to the meaning this word has in the logic of categorial objects – is more general, since it does not exclude transcendental beings from the field of the denoted relation.

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MIECZYSŁAW OMYŁA

NON-FREGEAN SEMANTICS FOR SENTENCES

1. INTRODUCTION

In this paper I intend to present the general and formal principles of non-Fregean semantics for sentences and to derive the simplest consequences of these principles. The semantic principles constitue foundation of non-Fregean sentential calculus and its formal semantics and the philosophical interpretations of it. Non-Fregean sentential calculus is the basic part of non-Fregean logic. Non-Fregean logic is a generalization of classical logic. It was conceived by Roman Suszko under the influence of Wittgensteinian's Tractatus Logico-Philosophicus. The term "non-Fregean" indicates that the set of semantic correlate of sentences need not contain of just two elements, as it assumed by Frege in [1]. Frege accepted the following semantic principle:

(A.F.) all true sentences have the same common referent, and similarly all false sentences also have the one common referent.

J. Łukasiewicz interpreted the common referent of true sentences as "Being" and analogically the common referent of all false sentences as "Unbeing". Suszko called the principle (A.F) the "semantical version of the Fregean axiom".

In [6] Suszko wrote: "If one accepts the Fregean Axiom then one is compelled to be an absolute monist in the sense that there exists only one and necessary fact".

According to Suszko (A.F.) has a counterpart in the language of classical logic which is a formula asserting that the universe of sentential variables is a two-element set. This formula is not expressed that fact in the language of non-Fregean logic.

In [5] Suszko presents the properties of his logic as follows: "... non-Fregean logic is the realization of the Fregean program in pure logic, logically bi-valent and extensional with two modifications: (1) keep formulas (sentences) and termes (names) as disjoint syntactic categories, having sense and denotations,as well, and (2) drop the desperate as-

Jan Woleński (ed.), Philosophical Logic in Poland, 153–165.

sumption that all true or false sentences have the same denotation (not sense, that is proposition)".

In the present paper the logical terminology has been modified so as to make it more natural to talk about the reference of sentences.

2. NON-FREGEAN FRAMEWORK FOR SENTENTIAL LANGUAGE

Let E be a non-empty set of arbitrary elements. The members of the set E will be called elementary sentences. Further, let $F = (f_1, f_2, \dots, f_n)$ be a finite sequence of arbitrary elements such that $E \cap F = \emptyset$ and let τ be an arbitrary function:

$$\tau : F \to N$$

where N is a set of natural numbers. The elements of F will be called connectives. The number of $\tau(f_i)$ will be denoted by v_i, and it will be said to be the number of arguments of the i-th connective.

The triple $< E, F, \tau >$ will be called an alphabet of the language considered. We define the set of sentences in this alphabet in the standard way:

DEFINITION 1. The set of sentences S in the alphabet $< E, F, \tau >$ is the smallest set among sets X such that

(i) $E \subseteq X$

(ii) for any $f_i \in F$: if $\alpha_1, \alpha_2, \dots, \alpha_{v_i} \in X$, then the sequence $f_i\alpha_1\alpha_2 \dots \alpha_{v_i}$ is the member of X.

In the set of sentences S the algebraic structure is given by treating every connective f_i as a v_i-argument operation defined in the set of sentences S as follows: for any $\alpha_1, \alpha_2, \dots, \alpha_{v_i} \in S : f_i(\alpha_1, \alpha_2, \dots, \alpha_{v_i}) = f_i\alpha_1\alpha_2 \dots \alpha_{v_i}$.

The abstract algebra $\boldsymbol{S} = (S, f_1, f_2, \dots, f_n)$ thus obtained will be called an algebra of sentences in the alphabet $< E, F, \tau >$.

To discuss about sentences from set S we use the following notation: The letters: $e, e_0, e_1, e_2, \dots$ will be used as variables ranging over the set of elementary sentences E; the letters $\alpha, \beta, \gamma, \dots$ will stand for arbitrary elements of the set S; the symbol $\alpha[e/\beta]$ will denote the result of substitution of each occurrence of elementary sentence e in sentence α by

sentence β; the symbols: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, will be the classical connectives of negation, conjunction, disjunction, implication and equivalence respectively. The letters X, Y, Z will represent arbitrary subsets of S, by Fin(X) we will mean the family of all finite subsets of set X and by $P(S)$ the family of all subsets of set S. Assume that on the collection $P(S)$ there is defined the operator:

$$Cn : P(S) \to P(S)$$

which for arbitrary $X, Y \in P(S)$ satisfies the conditions:

(i) $X \subseteq Cn(X)$
(ii) if $X \subseteq Y$ then $Cn(X) \subseteq Cn(Y)$
(iii) $Cn(Cn(X)) \subseteq Cn(X)$
(iv) if $Cn(X) = S$ then there exists $Y \in$ Fin(X) such that $Cn(Y) = S$.

Conditions (i)–(iii) assert that Cn is a consequence operator defined on the set S in the sense of Tarski. Condition (iv), based on Lindenbaum's lemma, ensures that for every Cn-consistent set of sentences X there exists a maximal Cn-consistent set Y such that $X \subseteq Y$.

DEFINITION 2. By a sentential language in the alphabet $< E, F, \tau >$ we shall mean any pair $(\boldsymbol{S}, Cn)$, where $\boldsymbol{S}$ is an algebra of sentences in alphabet $< E, F, \tau >$ and Cn is an arbitrary consequency operation on S satisfying conditions (i)–(iv).

To state the principles for interpreting any sentential language $L = (\boldsymbol{S}, Cn)$ the following set-theoretic constructions will be taken into consideration:

(1) An arbitrary non-empty set U. The elements of U will represent the situations. Intuitively speaking, the elements of U will be treated as given by coresponding sentences of L. For simplicity, elements of set U will be called situations. We use letters $p, q, r, \ldots$ as variables rangning the universe of situations U. The sentential variables: $p, q, r, \ldots$ are not neceserally in the alphabet of the language $\mathbb{L}$. However, the sentential variables are needed in the semantical metalanguage of $\mathbb{L}$, in which are formulated theorems about the set of semantic correlates of sentences of $\mathbb{L}$.

(2) An arbitrary relation H having as its domain the set S of sentences of L, and as its counter-domain the set of situations U, i.e. $H \subseteq S \times U$.

If $H(\alpha, p)$ holds we will say that the sentence α refers to the situation p, or a describes the situation p.

(3) The pair of sets (V, V_1) such that $\emptyset \neq V_1 \subseteq V$. The elements of V will be called logical values, and elements of V_1 the designated logical values.

(4) An any function:

$$v : S \to V$$

such that $\emptyset \neq v^{-1}(V_1) \neq S$. The function v will be called a logical valuation of $\mathbb{L}$. The family of sets $\{v^{-1}(V - V_1), v^{-1}(V_1)\}$ is called the fundamental partition of the set of sentences determined by the logical valuation v. Now we are ready to adopt the definition:

DEFINITION 3. An arbitrary sequence of the form:

$$(*) \quad < (S, Cn), U, (V, V_1), H, v >$$

is a semantic Suszko-framework iff it satisfies the following seven conditions, called the principles of non-Fregean semantics for sentences:

(P1) Principle of Logical Bivalence:
$V = \{1, 0\}, V_1 = \{1\}$, where 1, 0 are arbitrary objects.

(P2) Principle of the Maximality of Truth:
For any $X \subseteq S$, if $v^{-1}(V_1) = X$, then $Cn(X) \neq S$ and for any $\alpha \notin X$, $Cn(Xu\{\alpha\}) = S$.

(P3) Principle of Correlation:
For each sentence a of language $\mathbb{L}$ there is at least one situation $p \in U$ such that $H(\alpha, p)$.

(P4) Principle of Univocality:
For each sentence α of language $\mathbb{L}$ and for any situations $p, q \in U$, if $H(\alpha, p)$ and $H(\alpha, q)$ then $p = q$.

(P5) Principle of Stability:
For arbitrary $\alpha, \beta \in S$, if for any situation $p \in U$, $(H(\alpha, p) \Leftrightarrow H(\beta, p))$ then for each sentence $\gamma \in S$ and each elementary sentence e and any situation q, $(H(\gamma[e/\alpha], q) \Leftrightarrow (H(\gamma[e/\beta], q))$.

(P6) Principle of Subordination to Fundamental Partition:
For arbitrary sentences $\alpha, \beta \in S$, if for any situation p $(H(\alpha, p) \Leftrightarrow H(\beta, p))$, then $v(\alpha) = v(\beta)$.

(P7) Principle of Contextual Differentiation:
For arbitrary sentences $\alpha, \beta \in S$, if there exists a situation p such that if $\neg(H(\alpha, p) \Leftrightarrow H(\beta, p))$ then there exist $\gamma \in S$,

$e \in E$ such that $v(\gamma[e/\alpha] \neq v(\gamma[e/\beta])$.

The principles (P1) and (P2) pertain exclusively to relations between sentences and their truth values in the semantic framework considered. Elements of the set $v^{-1}(V_1)$ will be called true sentences of language $\mathbb{L}$ in the semantic framework (*), while elements of set $v^{-1}(V - V_1)$ will be referred to as false sentences in this framework. From the principle (P2) it follows that the set of true sentences in any semantic framework is a complete theory in the language $\mathbb{L}$. The distinction of sets U and V in semantic framework reflects the basic intuition underlying non-Fregean logic from the situation represented by a sentence is something different from the truth value of this sentence. Principles (P3), (P4), and (P5) deal solely with relations between sentences and situations. Observe that they do not contain any terms pertaining to sentences only, like those of a truth-value, of a consequence operator, or of entailment, etc. For this reason, each of the principles (P3)–(P5) has its own direct counterpart, formally identical, in the semantics of names. This indicates an analogy between the semantics of sentences and the semantics of names, an analogy frequently stressed by Suszko. Moreover, the three principles mentioned derive from the semantics of Frege. For, according to Frege,any correct symbolism should be such that every constant expression – i.e. one containing no free variables – has exactly one determinate sense which in turn determines exactly one reference for it. The principle of stability (P5) may be put also as follows: if two sentences α, β of $\mathbb{L}$ refer to the same situation, then exchanging them in an arbitrary sentence γ of $\mathbb{L}$ yields a sentence γ' which refers to the same situation as the original sentence γ.

The principle (P6) and (P7) deal with relations between truth-values of sentences and the situations presented by them. The principle (P6) says that if two sentences have different truth-values, then there must be at least one situation such that one of the two sentences refers to it while the other does not. And the principle (P7) states that if for any two sentences α, β of $\mathbb{L}$ there is a situation such that one of the two sentences refers to it while the other does not, then there must be in $\mathbb{L}$ at least one sentential context differentiating them with regard to their truth-values, i.e. a sentence γ such that $v(\gamma) \neq v(\gamma[\alpha/\beta])$.

Among Suszko's semantic frameworks we distinguish certain particular kinds of frameworks in which additional semantic principles

expressing certain ontological views are satisfied.

DEFINITION 4. A Suszko semantic framework will be called a Wittgensteinian framework iff the following condition obtains:

(PW) The Wittgesteinian Principle:
For any $\alpha, \beta \in S$, if $Cn(\alpha) = Cn(\beta)$ then for each situation $p \in U$, $H(\alpha, p) \Leftrightarrow H(\beta, p)$.

The principle (PW) is a reflection of Wittgensteinain's view expressed in Tractatus Logico-Philosophicus whereby logically equivalent sentences present the same situation.

DEFINITION 5. The Suszko semantic framework will be called a Fregean-framework iff the following principle holds:

(PF) The Fregean Principle:
For any sentences $\alpha, \beta \in S$, if $v(\alpha) = v(\beta)$ then for each situation $p \in U$, $H(\alpha, p) \Leftrightarrow H(\beta, p)$.

The principle (PF) states that any two sentences of the given language having the same truth value represent the same situation.

OBSERVATIONS: 1. The principles (P3) and (P4) taken together make it possible to define the function

$$h : S \rightarrow U$$

as follows: for each $\alpha \in S$ and any $p \in U$, $(p = h(\alpha) \Leftrightarrow H(\alpha, p))$. For the sake of simplicity we shall read the expression $p = h(\alpha)$ as 'p is the references of the sentence α'.

2. Principles (P1) and (P2) imply that logical valuations are the characteristic functions of complete theories in $\mathbb{L}$.

3. Choosing any particular set V among other two-element sets has no effect upon the reference of sentences in a Suszko-framework, nor upon the fundamental partition of the set of sentences as interpreted in that framework.

These observations allow us to simplify the notion of a Suszko semantic framework; namely, henceforth, any quadruple

$$(\mathbb{L}, U, h, v)$$

such that $\mathbb{L}$ is a sentential language, U is an arbitrary set, v is a characterictic function of an arbitrary complete theory in $\mathbb{L}$, and $h : S \rightarrow U$ is

a function such that for arbitrary sentences $\alpha, \beta \in S$ the following three conditions are satisfied:

(P5') if $h(\alpha) = h(\beta)$ then $\forall_{\gamma \in S} \forall_{e \in E}(h(\gamma[e/\alpha]) = h(\gamma[e/\beta]))$

(P6') if $h(\alpha) = h(\beta)$ then $v(\alpha) = v(\beta)$

(P7') if $h(\alpha) \neq h(\beta)$ then $\exists_{\gamma \in S} \exists_{e \in E}(v(\gamma[e/\alpha]) \neq v(\gamma[e/\beta]))$

will also be called a Suszko semantic framework or a non-Fregean semantics framework for sentences of language of $\mathbb{L}$ or in brief a non-Fregean semantics framework. The class of all non-Fregean frameworks will be named non-Fregean semantics (or alternatively Suszko semantics) for sentential languages. And the class of those non-Fregean frameworks in which the language $\mathbb{L}$ is fixed is called a non-Fregean semantics for that language. The following theorem characterizing non-Fregean semantics framework for sentences of any language.

THEOREM 1. The quadruple $(\mathbb{L}, U, h, v)$ is a non-Fregean semantics framework iff the following conditions are satisfied: (1) $\mathbb{L} = (\boldsymbol{S}, Cn)$ is an arbitrary sentential language, (2) U is an arbitrary set of at least two elements; (3) v is a characteristic function of some complete theory in $\mathbb{L}$; h is a function

$$h : S \rightarrow U$$

such that for arbitrary sentences: $\alpha, \beta \in S$ the condition:

$$(c) \quad h(\alpha) = h(\beta) \Leftrightarrow \forall_{\gamma \in S} \forall_{e \in E}(v(\gamma[e/\alpha]) = v(\gamma[e/\beta]))$$

is satisfied.

Proof. Assume that (*) $(\mathbb{L}, U, h, v)$ is a non-Fregean semantics framework, and first assume that for certain $\alpha, \beta \in$S, condition

(4) $h(\alpha) = h(\beta)$ holds. We then obtain successively:
(5) $\forall_{\gamma \in S} \forall_{e \in E}(h(\gamma[e/\alpha]) = h(\gamma[e/\beta]))$, from (P5')
(6) $\forall_{\gamma \in S} \forall_{\gamma \in E}(v(\gamma[e/\alpha]) = v(\gamma[e/\beta]))$, from (5) and (P6').

Let us now assume the right-hand side of equivalence (c). Now, on the basis of (P7') we get $h(\alpha) = h(\beta)$. To prove the second part of the theorem, let us assume conditions (1)–(3);we shall demonstrate that (*) is a non-Fregean semantics framework. It will suffice to show that principle (P5') holds, since (P6') and (P7') result directly from (c). To provide an indirect proof, let us assume that principle (P5') is not

satisfied, i.e. that for certain $\alpha, \beta \in S$, $h(\alpha) = h(\beta)$ and for certain $\gamma \in S$ and $e \in E$,

$$h(\gamma[e/\alpha]) \neq h(\gamma[e/\beta]).$$

In view of (c) there thus exist $\delta \in S$ and $e_0 \in E$ such that

$$(7) \qquad v(\delta[e_0/\gamma[e/\alpha]) \neq v(\delta[e_0/\beta]).$$

Let $d^* = \delta[e_0/\gamma]$. From (7) it follows that

$$v(\delta^*[e/\alpha] \neq v(\delta^*[e/\beta]),$$

which contradicts the assumption that $h(\alpha) = h(\beta)$.

Theorem 1 provides the criterion of identification of sentence references when the truth values of all sentences of the given language in the considered semantic framework are known. For semantic framework (*) let $T = \{\alpha : v(\alpha) = 1\}$. We may write condition (c) as

$$(8) \qquad h(\alpha) = h(\beta) \Leftrightarrow \forall_{\gamma \in S} \forall_{e \in S} (\gamma[e/\alpha] \in T \Leftrightarrow \gamma[e/\beta] \in T)$$

In virtue of principle (P2), T is a complete theory in $\mathbb{L}$. Therefore, if a certain complete theory T of language $\mathbb{L}$ is established as a set of sentences which are true in some semantic framework, then it is known from (8) which sentences of the language have the same reference , and which have different ones. J. Łoś considered in [2] the languages of sentential logic and gave the following definition of the notion of the interchangeability of expressions with regard to a given logical system X: "α is interchangeable with β with regard to the system X" if and only if we have: $\gamma \in X$ iff $\gamma[\alpha/\beta] \in X$, for any γ, and with X being an arbitrary set of expressions closed under substitution. Using the terminology of Łoś, we read Theorem 1 as follows: two sentences of the language $\mathbb{L}$ have the same reference in a particular non-Fregean framework if and only if they are interchangeable with regard to the set of all true sentences in that framework.

3. APPLICATION OF THE PRINCIPLES OF NON-FREGEAN SEMANTICS FOR SENTENCES IN THE INTERPRETATION OF LANGUAGES

Let $S = (\boldsymbol{S}, f_1, f_2, \ldots, f_n)$ be an algebra of sentences of type $< v_1, v_2, \ldots, v_n >$ of language $\mathbb{L} = (\boldsymbol{S}, Cn)$. The question now arises about the way in which the logical syntax of language $\mathbb{L}$ and the consequence operator Cn defined in this language, determine the structure of the set of sentences references in an arbitrary non-Fregean semantic framework:

$$(*) \quad (\mathbb{L}, U, h, v)$$

through the semantic principles. Let A be the set of those situations from the universe U which are references of sentences of the language $\mathbb{L}$, i.e.

$$A = \{a \in U : h(\alpha) = a, \quad \text{for some } \alpha \in S\}.$$

It follows immediately from the stability principle (P5) that we have:

COROLLARY 1. In any non-Fregean framework $(\mathbb{L}, U, h, v)$, and for any connective f_i of the language $\mathbb{L}$, and any of its sentences, the following condition holds:

$$(1) \quad \text{if } h(\alpha_1) = h(\beta_1), h(\alpha_2) = h(\beta_2), \ldots, h(\alpha_{v_i}) = h(\beta_{v_i}),$$
$$\text{then } h(f_1(\alpha_1, \alpha_2, \ldots, \alpha_{v_i})) = h(f_i(\beta_1, \beta_2, \ldots, \beta_{v_i})).$$

Moreover, we have:

THEOREM 2. Let $\mathbb{L}$ be an arbitrary language. If $\boldsymbol{S}$ is its algebra of sentences, and A is the set of references for all sentences of $\mathbb{L}$ in the non-Fregean framework $(\mathbb{L}, U, h, v)$, then for any connective f_i there is exactly one function

$$F : A^{v_i} \rightarrow A$$

such that for any $\alpha_1, \alpha_2, \ldots, \alpha_{v_i} \in S$ the following holds:

$$F_i(h(\alpha_1), h(\alpha_2), \ldots, h(\alpha_{v_i})) = h(f_i(\alpha_1, \alpha_2, \ldots, \alpha_{v_i})).$$

Proof: Let (1) $a_1, a_2, \ldots, a_{v_i}$ be an arbitrary sequence of the elements of A. Since A is a set of references of sentences of language $\mathbb{L}$, then there exist sentences $\alpha_1, \alpha_2, \ldots, \alpha_{v_i} \in S$ such that $a_1 = h(\alpha_1), a_2 = h(\alpha_2), \ldots, a_{v_i} = h(\alpha_{v_i})$. We define a function F as follows:

(2) $F_i(a_1, a_2, \ldots, a_{v_i}) = h(f_i(\alpha_1, \alpha_2, \ldots, \alpha_{v_i}))$.

The uniqueness of function F results from Corollary 1, since if

$$(3) \quad \begin{aligned} &h(\alpha_1) = h(\beta_1) = a_1 \\ &h(\alpha_2) = h(\beta_2) = a_2 \\ &\vdots \\ &h(\alpha_{v_i}) = h(\beta_{v_i}) = a_{v_i} \end{aligned}$$

then from (2), (3) and Corollary 1 we get

$$\begin{aligned} F_i(a_1, a_2, \ldots, a_{v_i}) &= h(f_i(\alpha_1, \alpha_2, \ldots, \alpha_{v_i})) \\ &= h(f_i(\beta_1, \beta_2, \ldots, \beta_{v_i})). \end{aligned}$$

DEFINITION 6. The function $F_i : A^{v_i} \to A$ where A is a set of references of
sentences in the semantic framework (*), will be called the reference of connective f in the semantic framework (*) iff for arbitrary sentences: $\alpha_1, \alpha_2, \ldots, \alpha_{v_i} \in S$ the following condition

(h) $F_i(h(\alpha_1), h(\alpha_2), \ldots, h(\alpha_{v_i})) = h(f_i(\alpha_1, \alpha_2, \ldots, \alpha_{v_i}))$

is satisfied. Hence we get the following corollary.

COROLLARY 2. In an arbitrary semantic framework (*) the logical syntax of language $\mathbb{L}$ imposes upon the set of references of the sentences of $\mathbb{L}$ the structure of an algebra similar to the algebra of sentences of $\mathbb{L}$.

Now we adopt the following definition of the notion an algebra of situations.

DEFINITION 7. The algebra $\boldsymbol{A} = (A, F_1, F_2, \ldots, F_n)$ will be called a situation algebra iff for some non-Fregean semantics framework (*) $(\mathbb{L}, U, h, v)$ the following two conditions are satisfied:

(i) A is a set of references of sentences of the language $\mathbb{L}$, and
(ii) $F_1, F_2, \ldots, F_n$ are references of connectives of language $\mathbb{L}$ in framework (*).

The situation algebra in semantic framework (*) will also be called an algebra associated with the framework (*) or algebra references of sentences of language $\mathbb{L}$ in framework (*). With regard to situation algebras we have:

COROLLARY 3. The function h correlating sentences with their references in non-Fregean framework $((\boldsymbol{S}, Cn), U, h, v)$ is an epimorphism of the algebra sentences $\boldsymbol{S}$ upon the situation algebra $\boldsymbol{A}$ with the universe A being a subset of the universe U of all situations.

COROLLARY 4. In any non-Fregean framework $(\mathbb{L}, U, h, v)$ the relation $\sim$ defined for any $\alpha, \beta \in S$ as: $\alpha \sim \beta$ iff $h(\alpha) = h(\beta)$ is a congruence of the algebra of sentences $\boldsymbol{S}$, and the corresponding situation algebra $\boldsymbol{A}$ is isomorphic to the quotient algebra $\boldsymbol{S}/\sim$.

COROLLARY 5. In every semantic framework $(\mathbb{L}, U, h, v)$ the function h is determined by the references of the elementary sentences of $\mathbb{L}$, and by the definition of the operations which constitute the reference for the connectives of $\mathbb{L}$ in that framework.

OBSERVATIONS: 1. In Theorem 1 and the Corollaries 1–4 we have to make use of the fact that the sentences of $\mathbb{L}$ form an algebraic structure, and also the principles (P3), (P4) and (P5).

2. In deriving Corollary 5, however, we have to make use also of the fact that the algebra of sentences $\boldsymbol{S}$ has as its set of generators the set of all elementary sentences of $\mathbb{L}$.

In interpreting the language $\mathbb{L}$ we now take into consideration the semantic principles pertaining to relations between references of sentences and the – truth values of these sentences. From principles (P1), (P2), (P6) taken together it follows that the set of references of sentences in an arbitrary semantic framework (*) splits into two disjoint subsets, namely into situations represented by true sentences and into situations represented by false sentences. This distinction leads to the following notion of a matrix associated with a semantic framework.

DEFINITION 8. The pair $M = (\boldsymbol{A}, D)$ will be called a matrix associated with the semantic framework (*) $(\mathbb{L}, U, h, v)$ – or alternatively – a

situation matrix iff the following conditions are satisfied:

(i) $\boldsymbol{A}$ is a situation algebra in (*),
(ii) $D = \{h(\alpha) : v(\alpha) = 1\}$.

Let (*) $(\mathbb{L}, U, h, v)$ be a non-Fregean semantic framework for a language $\mathbb{L} = (\boldsymbol{S}, Cn)$, and let $\boldsymbol{M} = (\boldsymbol{A}, D)$ be a situation matrix in that framework. The function $h : S \to U$, is a matrix epimorhism (m-epimorphism) of Lindenbaum matrix $(\boldsymbol{S}, T)$ on $\boldsymbol{M}$, where T is a complete theory of $\mathbb{L}$, i.e. h is an epimorphism of the algebra of sentences $\boldsymbol{S}$ on the an algebra of situations $\boldsymbol{A}$ and $h^{-1}(D) = T$. This epimorphism determines the congruence $\tilde{T}$ of the matrix $(\boldsymbol{S}, T)$ defined (in view of Theorem 1) as follows:

$$\alpha \underset{T}{\sim} \beta \text{ iff } \forall_{\gamma \in S}(\gamma \in T \text{ iff } \gamma[\alpha/\beta] \in T)$$

Thus the matrix $(\boldsymbol{A}, D)$ is isomorphic to the matrix $\mathbb{L}(T) = (\boldsymbol{S}/\!\sim T, T/\!\sim T)$, with T as the set of truth sentences in the considered framework (*). The matrix $\mathbb{L}(T)$ is known in logical literature as the Lindenbaum-Tarski matrix of the theory T, and it is defined as follows: $\boldsymbol{S}/_{\sim T} = (S/_{\sim T}, f_1^*, f_2^*, \ldots, f_n^*)$, with $S/_{\sim T} = \{|a| : \alpha \in S\}$, where $|a| = \{\beta \in S : \alpha \sim \beta\}$, and with $f_i^*(|\alpha_1|, |\alpha_2|, \ldots, |\alpha_{v_i}|) = |f(\alpha_1, \alpha_2, \ldots, \alpha_{v_i})|$, for any sentences in S. Moreover, $T/_{\sim T} = \{|a| : \alpha \in T\}$. Now it is easily seen that for any complete theory T of the language $\mathbb{L} = (\boldsymbol{S}, Cn)$ the sequence $(\mathbb{L}, S/_{\sim T}, h, v)$ with the function $h : S \to S/_{\sim T}$ defined as $h(\alpha) = |a|$, for any $\alpha \in S$, and with

$$v(\alpha) = \begin{cases} 1, & \text{if } \alpha \in T \\ 0, & \text{if } \alpha \notin T \end{cases}$$

is a semantic framework. And so we get

COROLLARY 6. A matrix $\boldsymbol{M} = (\boldsymbol{A}, D)$ is a situation matrix for the language $\mathbb{L}$ in some non-Fregean semantic framework if and only if the matrix $\boldsymbol{M}$ is isomorphic to the Lindenbaum-Tarski matrix of some complete theory in $\mathbb{L}$.
Proof. Let $\boldsymbol{M} = (\boldsymbol{A}, D)$ be a situation matrix in a non-Fregean semantic framework (*) $(\mathbb{L}, U, h, v)$ and let $h^{-1}(D) = T$.

From (P2) and (P6) it follows that T is a complete theory in $\mathbb{L}$. In virtue of previous considerations, the function

$$g : S/_{\sim T} \to A$$

defined for any $\alpha \in S$: $g(|a|) = h(\alpha)$, determines the isomorphism of matrices $\mathbb{L}(T)$ and $\boldsymbol{M}$.

Assume now that matrix $\boldsymbol{M} = (\boldsymbol{A}, D)$ is isomorphic with the Lindenbaum matrix $\mathbb{L}(T) = (\boldsymbol{S}/_{\sim T}, T/_{\sim T})$, where T is a certain complete theory in $\mathbb{L}$. We will prove that there exists a semantic framework such that M is a situation matrix in this framework. Indeed, the sequence $(\mathbb{L}, \boldsymbol{S}/_{\sim T}, k, v)$, with $k(\alpha) = |a|$, and

$$v(\alpha) = \begin{cases} 1, & \text{if } \alpha \in T \\ 0, & \text{if } \alpha \notin T \end{cases}$$

is a semantic framework. Now let the function $g : S/_{\sim T} \to A$ be an isorphism of matrices $\mathbb{L}(T)$ and $\boldsymbol{M}$. Then the sequence $(\mathbb{L}, U, h, v)$, where U is an arbitrary set such that $A \subseteq U$, $h = gk$, and v is a characteristic function of theory T, is a semantic framework such that M is the situation matrix in this framework.

Corollary 6 underlines the role of the Lindenbaum-Tarski matrices for complete systems in theory of references for sentences. This role was frequently stressed by Suszko.

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RELATIONAL SEMANTICS FOR NONCLASSICAL LOGICS: FORMULAS ARE RELATIONS

1. INTRODUCTION

Possible world semantics of logical languages introduced by Kanger and Kripke around 1960 is the most widely used technique for formal presentation of nonclassical logics. In spite of some shortcomings connected with the incompleteness phenomenon, it provides an intuitively clear interpretation of the fact that a formula is satisfied in some possible worlds and it might not be satisfied in some of the others. With this interpretation formulas can be treated as those subsets of a universe of possible worlds in which they are true. In most of the possible world models the truth conditions for the intensional propositional operations are articulated in terms of properties of a binary or ternary accessibility relation between possible worlds. It follows that from a formal point of view possible world semantical structures are not uniform. The part responsible for the extensional fragment of a logic under consideration determines a Boolean algebra of sets, and the part responsible for the intensional fragment refers to an algebra of relations. Our main objective in the present paper is to develop a unifying algebraic treatment of both extensional and intensional parts of logical systems.

The idea of relational semantics came from the interpretability of various nonclassical logics in some relational logics (Orłowska 1988, 1991, 1992a, 1992b). In those papers a construction is given of a validity-preserving embedding of formulas into relational terms. From the algebraic perspective this construction enables us to embed sets (unary relations) into relations of any higher rank. Under that embedding both formulas, formerly understood as sets of possible worlds, and accessibility relations receive a uniform representation as relations of the same finite rank. Both extensional and intensional propositional operations become nonclassical relational operations. The algebras obtained in this way are called nonclassical algebras of relations. Consider a simple example. Since in modal logics accessibility relations are binary

Jan Woleński (ed.), Philosophical Logic in Poland, 167–186.

relations, in the relational semantics modal formulas will be interpreted as binary relations as well. Let W be a universe of possible worlds. Given a formula A, its meaning is a binary relation R_A whose domain consists of those possible worlds at which A is true, and its range is the whole set W. Observe, that both domain and range of R_A carry an important information. Defining a concept, for example the concept 'even number', we normally say that an even number is a natural number which satisfies some condition. This definition consists of the two parts. The first part 'natural number' refers to a broader concept whose instances form a universe of instances which the instances of our concept 'even number' come from. The second part of the definition describes the positive instances of our concept. The relational interpretation of formulas is designed exactly according to that scheme. The domain of R_A consists of the positive instances of a concept described by A, and the range of R_A informs us about the admitted universe of objects to which our concept applies.

The main advantage of the relational semantics is the possibility of using the relational proof systems as the deductive systems for nonclassical logics. It enables us also to axiomatize various properties of accessibility relations, like irreflexivity, asymmetry, intransitivity. Within the relational framework we can easily axiomatize the extended multirelational versions of the known logics with the whole range of relation-algebraic operations on accessibility relations, including intersection and complement. Examples of relational proof systems can be found in Orłowska (1988, 1991, 1992a, 1992b). From the algebraic perspective the relational semantics enables us to expand applications of relation algebras to various nonclassical systems.

The paper is organized in four sections. In Section 2 we give a number of examples of intensional propositional operations. Many of them are taken from applied nonclassical logics inspired by some foundational problems in computer science and artificial intelligence. We mention, among others, the dynamic and specification operators introduced in the theory of programs, and the information operators developed in connection with knowledge representation in information systems. We also recall the intensional operators from the well-known logics such as modal logic, temporal logics, intuitionistic logic, Post many-valued logic, logics of relevance. In Section 3 we define several classes of nonclassical algebras of relations with the relational operations which are counterparts of the intensional logical operations. In Section 4

we define relational frames and semantics based on those frames for nonclassical logics.

2. NONCLASSICAL PROPOSITIONAL OPERATORS

Preliminaries. In this section a survey of propositional operators is given from various nonclassical logics defined by means of their interpretation in possible world models. Next, the operators will receive their relational counterparts in Section 3. Possible world models are usually determined by frames. In its standard form a frame is a relational system $F = (W, R)$ where W is a nonempty set of possible worlds and $R \subseteq W \times W$ is a binary accessibility relation between worlds. Frames of that form determine semantical structures for propositional languages, namely with each frame we associate models $M = (W, R, m)$ where m is a meaning function that assigns subsets of W to propositional variables. Then semantics is defined by means of the notion of satisfiability of formulas at worlds in a model. The inductive definition of satisfiability describes the truth conditions depending on the complexity of formulas, and for atomic formulas (propositional variables) we have:

(at) M, w sat p iff $w \in m(p)$ for any propositional variable p.

For formulas built with extensional operators such as classical negation, disjunction, conjunction or implication, their satisfiability at a possible world is completely determined by satisfiability of the subformulas of a given formula at that world:

($\neg$) M, w sat $\neg A$ iff not M, w sat A

($\vee$) M, w sat $A \vee B$ iff M, w sat A or M, w sat B

($\wedge$) M, w sat $A \wedge B$ iff M, w sat A and M, w sat B

($\rightarrow$) M, w sat $A \rightarrow B$ iff M, w sat $\neg A \vee B$.

For formulas built with intensional operators such as modal operators, temporal operators, Post many-valued operators, intuitionistic negation and implication, or some operators in relevant logics, their satisfiability at a world is defined not only in terms of satisfiability of their subformulas in that world but it also depends on a logical status of the subformulas in some worlds accessible from the current world. In the following a number of examples of those operators are given.

Having a definition of satisfiability, we can define truth and validity of formulas. A formula A is true in a model $M = (W, R, m)$ iff M, w sat A for all $w \in W$. A formula is true in a frame $F = (W, R)$ iff it is true in all models based on F. A formula is valid with respect to a class of frames iff it is true in all the frames from that class.

By an extension of a formula A in a model M we mean the set of those worlds from M where A is satisfied:

(ext) $\qquad \mathrm{ext}_M A = \{w \in W : M, w \text{ sat } A\}$.

It is easy to see that extensions of formulas built with the classical propositional operations are Boolean combinations of the extensions of their subformulas:

(ext $\neg$) $\qquad \mathrm{ext}_M \neg A = -\ \mathrm{ext}_M A$

(ext $\vee$) $\qquad \mathrm{ext}_M(A \vee B) = \mathrm{ext}_M \cup \mathrm{ext}_M B$

(ext $\wedge$) $\qquad \mathrm{ext}_M(A \wedge B) = \mathrm{ext}_M A \cap \mathrm{ext}_M B$

(ext $\rightarrow$) $\qquad \mathrm{ext}_M(A \rightarrow B) = -\mathrm{ext}_M A \cup \mathrm{ext}_M B$.

Formulas A and B are said to be equivalent ($A \equiv B$) iff for every model M and every world w we have M, w sat A iff M, w sat B.

Modal operators. Operators $[R]$, $<R>$ of necessity and possibility, respectively, determined by an accessibility relation are semantically defined as:

$[R]$ $\qquad M, w$ sat $[R]A$ iff for all $u \in W$ $(w, u) \in R$ implies M, u sat A

$<R>$ $\qquad M, w$ sat $<R>A$ iff there is an $u \in W$ such that $(w, u) \in R$ and M, u sat A.

In various modal logics the accessibility relation R is assumed to satisfy some conditions, more exactly, if FRM(Conditions) is a class of frames where relation R satisfies the 'Conditions', then the set of all the formulas valid with respect to that class is said to be its logic L (Conditions). For example: K (no restriction on accessibility), T (reflexive), KB (symmetric), B (reflexive, symmetric), K4 (transitive), KB4 (symmetric, transitive), S4 (reflexive, transitive), S5 (reflexive, symmetric, transitive), S4.1 (reflexive, transitive, atomic), KD (serial), KDB (symmetric, serial), KD4 (transitive, serial), S4.3.1 (reflexive, transitive, euclidean, discrete), G (transitive, well-capped).

Dynamic operators. A multimodal frame is a relational system of the form $F = (W, \mathrm{REL})$ where REL is a family of binary accessibility relations in W. If REL is closed with respect to the relational operations of union ($\cup$), composition (;), and reflexive, transitive closure ($*$), then the modal operators determined by those relations are called dynamic operators. The intuitive interpretation of possible worlds and accessibility relations in models based on those frames comes from the theory of programs (Pratt 1976, Harel 1979). Possible worlds are interpreted as computation states and accessibility relations as computer programs, more exactly, every relation defines a relationship between input states and output states of a program. The union of relations corresponds to the nondeterministic choice of a sequence of computation states, the composition of relations is a formal realization of the sequential composition of programs, and the star operation is interpreted as iteration of a program. The other informal meaning of dynamic operators comes from action logics. Here accessibility relations are interepreted as actions, action $R \cup S$ performs one of R and S, action $R; S$ performs first R, then S, and action R^* performs R finitely many times (including zero) sequentially. Formulas built with dynamic operators are then interpreted as follows:

[R]A	says that all the computation states obtained by executing program R possess properties described by formula A
<R>A	says that there is a computation of program R which ends with a state satisfying A.

Specification operators. A multimodal frame $F = (W, \mathrm{REL})$ is a specification frame if set REL of accessibility relations is closed on union, composition and residuations (/, \) defined as follows:

(/) $(x, y) \in R/S$ if for all z $(z, x) \in S$ implies $(z, y) \in R$

(\) $(x, y) \in S\backslash R$ iff for all z $(y, z) \in S$ implies $(x, z) \in R$.

Residuations are informally interpreted as program specifications (Hoare and Jifeng 1986). R/S is the weakest postspecification of S to achieve R, that is the greatest relation Q such that $S; Q \subseteq R$. In other words, given a program S, in order to achieve program R we have to execute R/S after the execution of S. Similarly, $S\backslash R$ is the weakest prespecification of S to achieve R, that is the greatest relation Q such that $Q; S \subseteq R$. Given a program S, to achieve R we have to execute

$S\backslash R$ before executing S. Dynamic logic with program specifications has been introduced in Orłowska (1989, 1992a). With the above interpretation the intuitive meaning of formulas built with modal operators determined by specifications is:

[R/S]A says that every state which can be obtained by meeting a postspecification of S to achieve R possesses properties described by formula A, and similarly for the possibility operator.

Difference operator. If in a multimodal frame set REL includes the diversity relation 0' $= \{(x, y) : x \neq y\}$, then in the corresponding modal logic we have the difference operator $<0'>$:

($<0'>$) M, w sat $<0'>A$ iff there is u such that $w \neq u$ and M, u sat A.

It follows that for every model M we have:

(ext$<0'>$) $\text{ext}_M <0'> A = \emptyset$ if $\text{ext}_M A = \emptyset$
$\text{ext}_M <0'> A = \text{ext}_M \neg A$ if $\text{ext}_M A$ is a singleton set
$\text{ext}_M <0'> A = W$ in the remaining cases.

The difference operator enables us to define propositional constants interpreted as single possible worlds, namely if p is a fixed propositional variable, then:

$$M, w \text{ sat } p \wedge \neg <0'> p \text{ iff } m(p) = \{w\}.$$

The other useful property expressible in the presence of the difference operator is:

$<0'><0'>A \leftrightarrow A$ is true in a model M iff the set of possible worlds of M consists of exactly two elements.

Humberstone operators. In Humberstone (1983) the modal operators for inaccessible worlds have been defined. Assume that we are given a bimodal frame of the form $F = (W, R, -R)$ including an accessibility relation and its complement. Then for any model based on such a frame the modal operators determined by $-R$ enable us to express properties of worlds which are not accessible from one another:

([-R]) M, w sat $[-R]A$ iff for all $u \in W$ if $(w, u) \notin R$ then M, u sat A.

(<-R>) M, w sat $<-R>A$ iff there is $u \in W$ such that $(w, u) \notin R$ and M, u sat A.

Sufficiency operators. Let a frame $F = (W, R)$ be given. Sufficiency operators $[[R]]$ are defined as follows (Gargov et al 1986):

([[R]]) M, w sat $[[R]]A$ iff for all $u \in W$ if M, u sat A, then $(w, u) \in R$.

It is easy to see that the Humberstone operators of inaccessible words can be defined by means of the sufficiency operators:

$$[[R]]A \equiv [-R]\neg A.$$

In multimodal logics with sufficiency operators we can axiomatize intersection of accessibility relations:

$$(\cap) \quad [[R \cap S]]A \leftrightarrow [[R]]A \wedge [[S]]A.$$

The operators provide also a means to axiomatize irreflexivity, asymmetry and intransitivity of relations in a frame, that is a relation R in a set W is irreflexive, asymmetric, or intransitive iff the respective formula is true in frame (W, R):

(irr) $[[R]]\neg A \to A$

(asym) $A \to [R]\neg[[R]]A$

(intran) $[[R]]\neg A \to [R][R]A$.

Temporal operators. To get access to both past and future worlds which are accessible from a present world we consider frames of the form $F = (W, R, R^{-1})$ including a relation and its converse. Modal operators determined by R refer to future states and those determined by its converse refer to the past states. That is $[R]$, $<R>$, $[R^{-1}]$, $<R^{-1}>$ are interpreted as 'always in the future', 'sometime in the future', 'always in the past', and 'sometime in the past', respectively (Burgess 1979). Usually, it is assumed that R is at least reflexive and transitive. The worlds are not necessarily linearly ordered because we may imagine that at any given moment of time the future develops in different directions. The other important temporal operators are binary operators Until and Since:

(Until) M, w sat AUntil$B$ iff there is $t \in W$ such that $(w, t) \in R$ and M, t sat B and for all $u \in W$ if $(w, u) \in R$ and $(u, t) \in R$, then M, u sat A

(Since) M, w sat ASince$B$ iff there is $t \in W$ such that $(t, w) \in R$ and M, t sat B and for all $u \in W$ if $(t, u) \in R$ and $(u, w) \in R$, then M, u sat A.

Thus AUntil$B$ means that A will be true until the next occurrence of B, and ASince$B$ means that since B occurred, A is true. In the presence of Until we can define the operator Next:

(Next) M, w sat Next$A$ iff M, w sat 0Until$A$ where 0 is a proposotional constant interpreted as 'false'.

Hence formula Next$A$ is satisfied at a world w iff there is a world t which is an immediate successor of w (with respect to R) and A is satisfied at t.

The alternative way of defining the next-state operator is to introduce a successor function explicitly in frames. By a standard frame with a successor we mean a system $F = (W, R, s)$ where R is a reflexive, transitive, and weakly connected relation in W, and s is a function in W which satisfies the following three conditions:

(s1) $(w, s(w)) \in R$
(s2) $(s(w), s(u)) \in R$ iff $(w, u) \in R$
(s3) The induction principle: For any $U \subseteq W$ if $w \in U$ and for all u $(w, u) \in R$ and $u \in R$ imply $s(u) \in U$, then for all t $(w, t) \in R$ implies $t \in U$.

Then we define the operator sNext as follows:

(sNext) M, w sat sNext$A$ iff $M, s(w)$ sat A.

Information operators. In recent applications of modal logics to knowledge representation, possible worlds and accessibility relations receive yet another interpretation different from their traditional logical and philosophical understanding. Instead of possible worlds the carrier of a frame consists of the objects whose properties are stored in an information system. Accessibility relations are defined in terms of a mapping f which assigns properties to those objects. Such an assignment is called an information function. Most often a property stored in an information system has a form of a pair consisting of an attribute

and its value. For example, if a system contains information that an object o is green, then we interpret this as the fact that $f(o$, colour) = green. However, a system might include also incomplete information about some properties of objects. For instance we might know the age of a person p only approximately, say between 25 and 30. In that case $f(p$, age) = $\{25, \ldots, 30\}$, that is f assigns a set of possible values of a given attribute to an object. In general, every information system is of the form $S = (OB, AT, \{VAL_a : a \in AT\}, f)$ where OB is a set of objects, AT is a set of attributes, VAL_a is a set of values of attribute a, and information function $f : OB \times AT \to \{VAL_a : a \in AT\}$ satisfies $f(o, a) \in VAL_a$ for every object o and attribute a. In frames determined by information systems we usually have the following accessibility relations (Orłowska 1985, Vakarelov 1989). Let A be a subset of attributes. An indiscernibility relation ind(A) and a similarity relation sim(A) are defined as:

(ind) $(w, u) \in \mathrm{ind}(A)$ iff $f(w, a) = f(u, a)$ for all $a \in A$

(sim) $(w, u) \in \mathrm{sim}(A)$ iff $f(w, a) \cap f(u, a) \neq \emptyset$ for all $a \in A$

Let $R(A)$ be anyone of the above relations. The following property of those relations constitute a basis of every axiomatization of information logics:

(R∪) $R(A \cup B) = R(A) \cap R(B)$ for any $A, B \subseteq$ AT.

The above condition says that having more attributes we can possibly make the respective relation 'finer' than the relations determined by subsets of those attributes. In other words the more properties we use to classify objects, the fewer objects will be indiscernible or similar, as a consequence they will be characterized more adequately in a given information system.

Given a set X of objects, its lower approximation $L(A)X$ is the set of those equivalence classes of relation ind(A) which are included in X, and its upper approximation $U(A)X$ is the set of those equivalence classes of ind(A) which have an element in common with X (Pawlak 1991). The modal operators determined by the accessibility relation ind(A) coincide with the operations of taking approximation, more precisely for every model M we have:

(ext[ind(A)]) $\mathrm{ext}_M[\mathrm{ind}(A)]B = L(A)\mathrm{ext}_M B$

(ext<ind(A)>) $\mathrm{ext}_M{<}\mathrm{ind}(A){>}B = u(A)\mathrm{ext}_M B.$

Modal operators determined by a similarity relation sim(A) can be treated as generalized approximation operations whose definitions are obtained from the definitions of approximations through replacement of equivalence classes by the similarity classes of the given relation.

Intuitionistic operators. Let a frame $F = (W, R)$ be given such that R is a reflexive and transitive relation in W (Kripke 1965). We consider models $M = (W, R, m)$ based on F such that the meaning function m satisfies an atomic heredity condition:

(h) If $(w, u) \in R$ and $w \in m(p)$, then $u \in m(p)$ for an propositional variable p.

It is well known that condition (h) extends to all the formulas. The intensional operators of intuitionistic negation and implication are then defined as follows:

($\neg_{\mathrm{int}}$) M, w sat $\neg_{\mathrm{int}}A$ iff for all $u \in W$ if $(w, u) \in R$, then not M, u sat A

($\rightarrow_{\mathrm{int}}$) M, w sat $A \rightarrow_{\mathrm{int}} B$ iff for all $u \in W$ if $(w, u) \in R$ and M, u sat A, then M, u sat B.

Many-valued operators. In many-valued Post logic the intensional operators are those of intuitionistic negation and implication, and moreover unary operators D_i for $1 \leq i < \omega$ and propositional constants E_j for $0 \leq j \leq \omega$ which can be treated as nullary operators. We recall their possible world semantics developed in Maximova and Vakarelov (1974). By a Post frame we mean a frame $F = (W, R, \{d_i : 1 \leq i < \omega\})$, where each d_i is a function in W. Every Post frame is assumed to satisfy the following conditions for any $i, j = 1, 2, \ldots$:

(P1) R is reflexive and transitive
(P2) $d_i d_j x = d_i x$
(P3) $(x, d_1 X) \in R$
(P4) $(d_i +_1 x, d_i x) \in R$
(P5) $(x, y) \in R$ implies $(d_i x, d_i y) \in R$
(P6) $(d_i x, x) \in R$ iff $(x, d_i +_1 x) \notin R$
(P7) For any $x \in W$ there is i such that $(d_i x, x) \in R$.

Models based on Post frames are systems of the form $M = (W, R, \{d_i : 1 \leq i < \omega\}, m)$ where meaning function m satisfies the atomic

heredity condition (h). The Post operators are defined as:

(D_i) M, w sat $D_i A$ iff $M, d_i w$ sat A for $1 \leq i < \omega$

(E_j) M, w sat E_j iff $(d_j w, w) \in R$ for $0 < j < \omega$, $E_0 = \emptyset$, $E_\omega = W$.

For a detailed presentation of the logic see Post (1921), Epstein (1960), Rasiowa (1973).

Relevant operators. Possible world semantics for relevant logics have been developed in Urquhart (1972) and Routley and Meyer (1973). Frames for relevant logics are systems of the form $F = (W, R, 0, *)$ where $R \subseteq W^3$ is a ternary accessibility relation that provides interpretation of implication and intensional conjunction, 0 is a distinguished world called the real world, and $* : W \to W$ is a function in W that provides interpretation of negation. It is assumed that $R, 0, *$ satisfy the following conditions:

(R1) $R0aa$
(R2) $R0ab$ and $Rbtu$ imply $Ratu$
(R3) $Rabc$ implies Rac^*b^*
(R4) $a^{**} = a$.

Models of relevant logics are systems of the form $M = (W, R, 0, *, m)$ where the meaning function m satisfies the following form of the atomic heredity condition:

(R5) $R0ab$ and $a \in m(p)$ imply $b \in m(p)$ for any propositional variable p.

Then the intensional operators of relevant negation, implication and fusion are defined as follows:

$(\neg_{\text{rel}})$ M, w sat $\neg_{\text{rel}} A$ iff not M, w^* sat A

$(\to_{\text{rel}})$ M, w sat $A \to_{\text{rel}} B$ iff for all $x, y \in W$ if $Rwxy$ and M, x sat A, then M, y sat B

(o) M, w sat AB iff there are $x, y \in W$ such that $Rxyw$ and M, w sat A and M, y sat B.

3. NONCLASSICAL ALGEBRAS OF RELATIONS

Preliminaries. By the full algebra of binary relations over a set W we

mean the algebra:

$$Re(W) = (Sb(W \times W), -, \cup, \cap, 1, 0, ;, ^{-1}, 1')$$

where $(Sb(W \times W), -, \cup, \cap, 1, 0)$ is the Boolean algebra of all the subsets of $W \times W$, ; and $^{-1}$ are the relational composition and converse, respectively, and 1' is the identity in W.

The class *RRA* of representable relation algebras consists of isomorphic copies of subalgebras and direct products of full algebras of binary relations:

$$RRA = ISP\{Re(W) : W \text{ is a set}\}.$$

Every algebra from *RRA* is isomorphic to an algebra whose elements are binary relations, 1 is an equivalence relation, and 1' is the identity on the field of 1. If 1 has exactly one equivalence class, then such an algebra is said to be simple. The class *RRA* is a variety (Tarski 1941, 1955), but its equational theory is not finitely axiomatizable (Monk 1964, 1969). Moreover, an infinite axiomatization of *RRA* requires infinitely many relation variables.

In a general setting a relation algebra is an algebra of the form (Jónsson 1982):

$(A, -, +, \cdot, 1, 0, ;, ^{\cup}, 1')$ such that

(RA1) $(A, -, +, \cdot, 1, 0)$ is a Boolean algebra

(RA2) $(A, ;, ^{\cup}, 1')$ is a monoid with involution, that is
$x; (y; z) = (x; y); z$
$(x; y)^{\cup} = y^{\cup}; x^{\cup}$
$x^{\cup\cup} = x$
$x; 1' = x = 1'; x$

(RA3) The operations ; and $^{\cup}$ are distributive over +
$(x + y); z = x; z + y; z, \ x; (y + z) = x; y + x; z$
$(x + y)^{\cup} = x^{\cup} + y^{\cup}$

(RA4) $x^{\cup}; -(x; y) \leq -y$ where $\leq$ is the lattice ordering such that $u < v$ iff $u + v = v$ iff $u \cdot v = u$.

The class *RA* of relation algebras is a variety. Examples of algebras which are in *RA* but not in *RRA* can be found in Lyndon (1950). By right (/) and left (\) residual of x over y we mean the elements defined as follows:

(RA/) $\quad x/y = -(y^{\cup}; -x)$

(RA\) $\quad y\backslash x = -(-x; y^{\cup})$.

Let 0' denote the element $-1'$. The two operations of residuation can be taken as primitive in place of $^{\cup}$, which is then defined as

(RA$^{\cup}$) $\quad x^{\cup} = 0'/-x \;\; x^{\cup} = -x\backslash 0'$.

Logical relational operations. With every logical operation presented in Section 2 we associate a relational operation. These operations are called logical relational operations (lr-operations). For the sake of simplicity they are denoted with the same symbols as in the logical languages. Let a, b, r be binary relations. Clearly, the relational counterparts of the classical propositional operators of negation, disjunction and conjunction are the Boolean operations of complement, union, and intersection, respectively. Then the classical relational implication is defined as:

(lr $\rightarrow$) $\quad a \rightarrow b = -a \cup b$.

The relational counterparts of the intensional logical operations are the following:

(lr[r]) $\quad [r]a = \{(w, z)$: for all u if $(w, u) \in r$, then $(u, z) \in a\}$

(lr$<r>$) $\quad <r>a = \{(w, z)$: there is u such that $(w, u) \in r$ and $(u, z) \in a\}$.

If r is a relational term built with ;, +, $*$, then we obtain the dynamic relational operators. If r is built with residuations, then we have the relational specification operators. The difference operator and Humberstone operators receive the following relational counterparts:

(lr$<0'>$) $\quad <0'>a = \{(w, z)$: there is u such that $(w, u) \in 0'$ and $(u, z) \in a\}$

(lr[$-r$]) $\quad [-r]a = \{(w, z)$: for all u if $(w, u) \in -r$, then $(u, z) \in a\}$

(lr$<-r>$) $\quad <-r>a = \{(w, z)$: there is u such that $(w, u) \in -r$ and $(u, z) \in a\}$

For sufficiency operators and temporal operators we have:

(lr$[[r]]$) $[[r]]a = \{(w,z)$: for all u if $(u,z) \in a$, then $(w,u) \in r\}$

(lrUntil) aUntil$b = \{(w,z)$: there is t such that $(w,t) \in r$ and $(t,z) \in b$ and for all u if $(w,u) \in r$ and $(u,t) \in r$, then $(u,z) \in a\}$

(lrSince) aSince$b = \{(w,z)$: there is t such that $(t,w) \in r$ and $(t,z) \in b$ and for all u if $(t,u) \in r$ and $(u,w) \in r$, then $(u,z) \in a\}$

(lrNext) Next$a = \{(w,z)$: there is t such that $(w,t) \in r$ and $(t,z) \in a$ and there is no u such that $(w,u) \in r$ and $(u,t) \in r\}$.

Let r be a reflexive, transitive and weakly connected relation. We define a relational sNext operator determined by a successor relation s:

(lrsNext) sNext$a = \{(w,z) : (s(w),z) \in a\}$ where s is a relation which satisfies:

(lrs0) $s;s^{\cup} \leq 1'$

(lrs1) $1' \leq s;r^{\cup}$

(lrs2) $s;r;s^{\cup} = r$

(lrs3) For any a if $(w,z) \in a$ and for all u $(w,u) \in r$ and $(u,z) \in a$ imply $(s(u),z) \in a$, then for all t $(w,t) \in r$ implies $(t,z) \in a$.

The first condition above says that relation s is a function, the remaining ones are direct counterparts of the respective conditions postulated for a successor function in possible world frames.

In a natural way we obtain intuitionistic relational operators:

(lr $\neg_{\text{int}}$) $\neg_{\text{int}}a = \{(w,z)$: for all u if $(w,u) \in r$, then $(u,z) \notin a\}$

(lr $\rightarrow_{\text{int}}$) $a \rightarrow_{\text{int}} b = \{(w,z)$: for all u if $(w,u) \in r$ and $(u,z) \in a$, then $(u,z) \in b\}$ where relation r is reflective, transitive and, moreover, it satisfies the counterpart of the heredity condition:

(lrh) If $(w,u) \in r$ and $(w,z) \in a$, then $(u,z) \in a$ for any a.

Clearly, these conditions can be written in the relational form: $1' \leq r$, $r;r \leq r$, for all $ar^{\cup};a \leq a$.

To define Post many-valued relational operators we assume that r is as above and, moreover, we are given a family of binary relations d_i for $i = 1, 2, \ldots$ which satisfy the following conditions:

(lrP2) $\quad d_j; d_i \leq d_i$

(lrP3) $\quad 1' \leq d_i; r^{\cup}$

(lrP4) $\quad 1' \leq d_{i+1}; r; d_i^{\cup}$

(lrP5) $\quad r \leq d_i; r; d_i^{\cup}$

(lrP6) $\quad 1' \leq d_i; r$ iff not $1' \leq d_{i+1}; r^{\cup}$

(lrP7) $\quad$ There is $i = 1, 2, \ldots$ such that $1' \leq d_i; r$.

Then we define:

(lrD_i) $\quad D_i a = \{(w, z) : (d_i w, z) \in a\}$

(lrE_j) $\quad E + j = \{(w, z) : (d_j, w, w) \in r\}$ for $0 < j < \omega$, $E_0 = 0$, $E_\omega = 1$.

Relevant algebras of relations are the algebras of ternary relations. Assume that we are given ternary relations 0, $*$, and r which satisfy the following conditions:

(lrR0) $\quad 0 = \{(w, z, t) : w = 0\}$, $* = \{(x, z, t) : w^* = z\}$

(lrR1) $\quad$ If $(w, z, t) \in r \cdot 0$, then $z = t$

(lrR2) $\quad$ If $(w, z, t) \in r \cdot 0$ and $(t, x, y) \in r$, then $(z, x, y) \in r$

(lrR3) $\quad$ If $(w, z, t) \in r$, then there are x, y such that $(t, x, u) \in *$ and $(z, y, u) \in *$ and $(w, x, y) \in r$

(lrR4) $\quad$ For all w, t there are x, y such that $(y, x, t) \in *$ and $(w, y, t) \in *$ iff $w = x$.

Now we define the relevant lr-operations:

(lr$\neg_{\text{rel}}$) $\neg_{\text{rel}} a = \{(w,z,t)$: there is x such that $(w,x,t) \in *$ and $(x,z,t) \notin a\}$

(lr$\rightarrow_{\text{rel}}$) $a \rightarrow_{\text{rel}} b = \{(w,z,t)$: for all x,y if $(w,x,y) \in r$ and $(x,z,t) \in a$, then $(y,z,t) \in b\}$

(lro) $ab = \{(w,z,t)$: there are x,y such that $(x,y,w) \in r$ and $(x,z,t) \in a$ and $(y,z,t) \in b\}$.

Some of the lr-operations are definable in terms of the standard relation-algebraic operators. It follows that the respective classes of nonclassical algebras of relations are generalized reducts of *RRA*, that is for every operator in a given class of nonclassical algebras of relations there is a term function over *RRA* which defines it. We have:

$$<r>a = r;a$$

$$[r]a = a/r^{\cup} = -(r;-a)$$

$$<0'>a = -1';a$$

$$[[r]]a = -(-r;-a)$$

$$\text{Next}a = r;a \cdot -(r;r)$$

$$\text{sNext}a = s;a$$

$$a \rightarrow_{\text{int}} b = -(r;(a \cdot -b))$$

$$\neg_{\text{int}} a = -(r;a)$$

$$D_i a = d_i;a$$

The relevant lr-operators as well as the respective conditions on 0, $*$, r are definable in terms of operators from algebras of relations of the rank higher than 2 (cylindric algebras), however, we omit the technical details here.

Ideal relations. An element x of a relation algebra is said to be a (right) ideal element whenever $x;1 = x$. It is easy to see that if x and y are ideal elements, then so are $-x$, $x+y$, $x \cdot y$, $x \rightarrow y$. Moreover, if y is an ideal element, then for any x the elements $x;y$ and y/x are ideal too. As a consequence, if a,b are ideal elements, then so are $<r>a$, $[r]a$, sNext$a$, $a \rightarrow_{\text{int}} b$, $\neg_{\text{int}} a$, $d_i a$.

Ideal binary relations in a set W are of the form $U \times W$ for $U \subseteq W$. A ternary relation is ideal iff it has the form $U \times W \times W$. If a,b are ternary ideal relations, then so are $\neg_{\text{rel}} a$, $a \rightarrow_{\text{rel}} b$, and ab.

Ideal relations can be viewed as unary relations (or sets) which are 'dummy embedded' into relations of any higher rank. As a consequence class *RRA* is a generalized reduct of a class of subalgebras of algebras of relations of any rank higher than 2.

4. RELATIONAL SEMANTICS

In the relational semantics formulas of nonclassical logics are interpreted as ideal relations. The rank of those relations is the same as the rank of the accessibility relation in the possible world models of the given logic. In particular, formulas from modal logic, temporal logic, Post many-valued logic, and intuitionistic logic become binary ideal relations, and formulas from relevant logics become ternary ideal relations.

By a relational frame we mean a system of the form:

$$F = (W, R, \{R_p\}_{p \in \text{VARPROP}})$$

where W is a nonempty set, R is a relation in W of a finite rank k, and for any propositional variable p R_p is an ideal relation in W of rank k, that is R_p is of the form $X \times W^{k-1}$, where $X \subseteq W$. Relational models based on frame F are systems $M = (F, m)$ where m is a mapping that assigns ideal relations in W to formulas. For atomic formulas $m(p)$ equals R_p, and for formulas built with the classical propositional operators of negation, disjunction and conjunction, m assigns the relations which are built from the component relations with complement, union, and intersection, respectively. To formulas built with intensional n-argument connectives function m assigns the relations built with the respective lr-operations:

$$m(p) = R_p \text{ for any propositional variable } p$$

$$m(\neg A) = W^k - m(A),$$

$$m(A \vee B) = m(A) \cup m(B),$$

$$m(A \wedge B) = m(A) \cap m(B)$$

$$m(\text{operation}(A_1, \ldots A_n)) = \text{ lr-operation } (m(A_1), \ldots m(An))$$

A formula A is true in a relational model $M = (W, R, \{R_p\}_{p \in \text{VARPROP}, m})$ if $m(A)$ is the universal relation in W. For binary relations that requirement is $m(A) = W \times W$, and for ternary relations $m(A) = W \times W \times W$. A formula is valid in the relational semantics iff it is true in all the relational models.

Let a relational model M be given. We define a possible world model $M' = (W, R, m')$ such that set W and accessibility relation R are the same as in M, and meaning function m' assigns the domain of relation R_p to the propositional variable p:

$$m'(p) = \mathrm{dom} R_p \text{ for any propositional variable } p.$$

Any possible world model defined as above is called the model determined by M.

The following theorem establishes a relationship between the possible world semantics and the relational semantics.

Theorem. Let M be a relational model of a nonclassical logic L, and let M' be the possible world model determined by M. Let k be a rank of the accessibility relation in M. Then for any formula A of L we have:

(a) M', w sat A iff $(w, z_1, \ldots, z_{k-1}) \in m(A)$ for any $z_1, \ldots, z_{k-1} \in W$

(b) $\mathrm{ext}_{M'} A$ = domain of $m(A)$

(c) A is valid in the possible world semantics for L iff A is valid in the relational semantics for L.

The proof of (a) is by induction with respect to the complexity of formula A. As an example consider a formula from modal logic and from relevant logic. Assume that L is a modal logic and let A be of the form $<R>B$. We have M', w sat $<R>B$ iff there is $u \in W$ such that $(w, u) \in R$ and M', u sat B. By the induction hypothesis $(u, z) \in m(B)$ for any z. This means that $(w, z) \in R; m(B)$. As was shown in Section 3 it is just the result of applying to $m(A)$ the lr-operation corresponding to $<R>$, and hence $(w, z) \in m(A)$.

Now let L be a relevant logic, and consider formula $B \rightarrow_{\mathrm{rel}} C$. We have M', w sat $B \rightarrow_{\mathrm{rel}} C$ iff for all $x, y \in W$ if $(w, x, y) \in R$ and M', x sat B, then M', y sat C. By the induction hypothesis for all $x, y \in W$ if $(w, x, y) \in R$ and $(x, z, t) \in m(B)$, then $(y, z, t) \in m(C)$. We conclude that $(w, z, t) \in m(B) \rightarrow_{\mathrm{rel}} m(C) = m(B \rightarrow_{\mathrm{rel}} C)$.

Condition (b) follows directly from (a). To prove (c) observe that for every relational model M there is a possible world model N such that formula A is true in M iff A is true in N. Clearly it is sufficient to take M' as N. Conversely, for every possible world model N there is a relational model M such that A is true in N iff A is true in M. Let $N = (W, R, n)$ be given. Define $M = (W, R, \{R_p\}_{p \in \mathrm{VARPROP}, m})$ taking $R_p = n(p) \times W \times \ldots \times W$ of a sufficient rank. It is easy to

see that the models satisfy the required condition, which completes the proof.

The theorem shows that the nonclassical algebras of relations correspond adequately to various nonclassical logics. In a more general setting, any element r of a relation algebra (instead of the accessibility relation) determines a family of logical relational operations in a set of ideal elements of that algebra. The natural question arises about the mutual influence of properties of r and properties of the underlying algebra of ideal elements. This problem is in a sense parallel to the problem of correspondence in logic (Van Benthem 1984). In Orłowska (1990) the formulation of the problem of correspondence for modal relation algebras has been given and several examples of correspondences are presented. Equations which axiomatize the classes of nonclassical relation algebras corresponding to nonclassical logics are obtained directly from the axioms of those logics together with the respective conditions on the parameter element r determining the lr-operations. However, the further systematic study is needed of connections between nonclassical relation algebras determined by classes of relational elements and relation algebras themselves. On the one hand the results known from logic can be reformulated for the respective algebras, and on the other hand the method illustrated in the present paper offers a technique to engage the tools of relation algebras in one more field of logical investigations.

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JACEK PAŚNICZEK

NON-STANDARD POSSIBLE WORLDS, GENERALISED QUANTIFIERS, AND MODAL LOGIC

1. NON-STANDARD POSSIBLE WORLDS

The concept of non-standard possible worlds introduced by N. Rescher and R. Brandom in their book *The Logic of Inconsistency. A Study in Non-standard Possible-World Semantics and Ontology*[1] is a generalisation of the concept of "standard" possible worlds. A non-standard possible world (in short: n-world), in contrast to a standard possible world (in short: p-world), can be inconsistent and incomplete. An n-world is inconsistent if for some proposition A, both A and $\neg A$ obtain in the world; an n-world is incomplete if for some proposition A neither A nor $\neg A$ obtain in the world.[2]

Let us think of n-worlds as totally determined by sets of propositions obtaining in the worlds: there will be a one-to-one correspondence between n-worlds and sets of propositions. Some n-worlds may be viewed as p-worlds. i.e. those which are consistent and complete. But there is another way of starting with ontology of n-worlds. Let $\mathbb{W}$ be the set of all p-worlds. Traditionally, propositions are treated as subsets of $\mathbb{W}$ (we pass over some flaws of such treatment). Thus, n-worlds can be considered as correlated with subsets of $\mathcal{P}(\mathbb{W})$, where $\mathcal{P}(\mathbb{W})$ is the power set of $\mathbb{W}$. For simplicity, if there is no danger of ambiguity, we shall identify n-worlds with subsets of $\mathcal{P}(\mathbb{W})$.

The general principle of n-world ontology will be the following:

(1) A proposition A obtains in n-world S iff there exists a set $X \in S$ such that for every p-world $w \in X, A$ obtains in w.

Notice that the principle stipulate an obvious closure condition: if a proposition A is correlated with a set $X \subseteq \mathbb{W}$, then it obtains in every n-world S such that for some $Y \in S, Y \subseteq X$.

Let us consider some instances of n-worlds. Let $s, u, v, w \in \mathbb{W}$. According to the principle, exactly the same propositions obtain in p-world s and in n-world $\{\{s\}\}$. Thus we can interpret p-worlds as some n-worlds. Notice that a proposition obtains in n-world $\{\{s, w\}\}$

Jan Woleński (ed.), Philosophical Logic in Poland, 187–198.

iff it obtains both in s and in w; what follows then is that n-world is basically[3] incomplete. A proposition obtains in n-world $\{\{s\},\{w\}\}$ iff it obtains in s or in w; what follows then is that this n-world is basically inconsistent. A proposition obtains in n-world $\{\{s,u\},\{v,w\}\}$ iff it obtains both in s and u or both in v and w; what follows then is that this n-world is basically incomplete and inconsistent. All propositions obtaining in every p-world and only those obtain in n-world $\{\mathbb{W}\}$; we call that n-world the *universal* n-world. All propositions obtaining in some p-world and only those obtain in n-world $\mathcal{P}(\mathbb{W})-\{\emptyset\}$; we call that n-world the *existential* n-world. Every proposition obtains in n-world $\{\emptyset\}$, and no proposition obtains in n-world $\emptyset$. The ontological status of the last two n-worlds is somewhat controversial since, by contrast with other n-worlds, in $\{\emptyset\}$ even negations of logical laws obtain, and in $\emptyset$ no logical law obtains. For this reason, and to simplify further considerations, we exclude these limit n-worlds from our ontology, assuming that only non-empty subsets of $\mathcal{P}(\mathbb{W})-\{\emptyset\}$ represent n-worlds.

Our approach suggests that p-worlds are ontologically more primitive than n-worlds since the latter are set-theoretically built out of the former. But this is a matter of our formalisation of n-world ontology; this however need not be so. We may for instance choose an algebraic approach as Rescher and Brandom have done. They impose on n-worlds the distributive lattice structure with *join* and *meet* operations: $U \cup V$ is understood as the n-world in which everything obtains that obtains in U or in V, $U \cap V$ is understood as the n-world in which everything obtains that obtains in U and in V.[4] That lattice structure can be further enriched by assuming that it is a De Morgan lattice with minimum and maximum elements (the elements will correspond to the universal and existential n-worlds respectively).[5] However, our way of formalising n-world ontology is motivated by its relative simplicity and by its similarity to other formal notions which we are to discuss.

2. A LOGIC OF GENERALISED QUANTIFIERS

Now we will describe briefly a simple logic of unary generalised quantifiers which is a slight extension of classical first-order logic.[6]

Let L_Q be a language with the same alphabet as the classical language L of first-order logic, i.e. the alphabet of L_Q consists of connectives, classical quantifiers $(\forall, \exists)$, the identity symbol, predicate and function symbols, individual variables and constants, brackets. The grammar

of L_Q differs from the grammar of L only in that constants occupy quantifier positions and only those positions. More precisely, atomic formulae of L_Q are the same as in L, and if A, B are formulae, p is a classical quantifier or constant then the following expressions are formulae: $\neg A, (A \supset B), pxA$. (From now on, let p, q be a classical quantifier or constant, x, y be variables, A, B be formulae).

The semantics of L_Q resembles closely the semantics of L. Given any non-empty set $\mathbb{D}$, let I be a function which is defined on predicate and function symbols just like the ordinary interpretation function for L, and which assigns non-empty subsets of $\mathcal{P}(\mathbb{D}) - \{\emptyset\}$ to classical quantifiers and constants. In particular, $I(\forall) = \{\mathbb{D}\}$, $I(\exists) = \mathcal{P}(\mathbb{D}) - \{\emptyset\}$. The pair $<\mathbb{D}, I>$ is called the model of L_Q. The notion of variable assignment is the same as in the semantics for L. The only specific truth condition in the semantics of L_Q is that for formulae of the form 'pxA':

(2) With respect to a given assignment: the formula pxA is true in $<\mathbb{D}, I>$ iff there exists a set $X \in I(p)$ such that $X \subseteq \{d \in \mathbb{D} : A \text{ is true if } d \text{ is assigned to } x\}$.

It is easy to notice that the condition retains the meaning of the classical quantifiers.

The logic **Q** based on L_Q is axiomatised in the following way.

- $AQ1$ Classical truth-functional tautologies
- $AQ2$ $\forall x(A \supset B) \supset (pxA \supset pxB)$
- $AQ3$ $A \supset \forall xA$, provided x is not free in A
- $AQ4$ $\forall xA \supset A^*$, where A^* is obtained from A by freely substituting every free occurence of x by some variable
- $AQ5$ $pxA \supset pyA^*$, where A and A^* differ only in that A has free x where and only where A^* has free y
- $AQ6$ $x = x$
- $AQ7$ $x = y \supset (A \supset A^*)$, where A and A^* differ only in that in one or more places where A has free x, A^* has free y
- MP if $\vdash_Q A \supset B$ and $\vdash_Q A$, then $\vdash_Q B$
- QG if $\vdash_Q A$, then $\vdash_Q pxA$ and $\vdash_Q \neg px\neg A$

The logic **Q** is an extension of classical first order logic and indeed an essential extension. Let us list some theorems of **Q**:

$TQ1$ $\forall xA \supset pxA$

$TQ2$ $pxA \supset \exists xA$

$TQ3$ $px(pxA \supset A)$

$TQ4$ $px(A \wedge B) \supset (pxA \wedge pxB)$

$TQ5$ $px\neg py(x \neq y) \supset ((pxA \wedge pxB) \supset px(A \wedge B))$

$TQ6$ $\exists x\neg py(x \neq y) \supset (px\neg A \supset \neg pxA)$

$TQ7$ $\exists xpy(x = y) \supset (\neg pxA \supset px\neg A)$

$TQ8$ $px\neg py(x \neq y) \supset (px(A \supset B) \supset (pxA \supset pxB))$

$TQ9$ $\neg px\neg qy(x = y) \supset (pxA \supset qxA)$

$TQ10$ $px\neg qy(x \neq y) \supset (qxA \supset pxA)$

Completeness, compactness, and Skolem-Löwenheim theorems hold in **Q**. It is worth emphasising that the completeness of **Q** easily follows from the completeness of the classical first-order logic.[7] Notice that **Q** is the logic for monotone increasing quantifiers only (see $AQ2$).

We should point to some important feature of **Q**. Suppose that p is a constant such that the formula $\exists x\neg py(x \neq y) \wedge \exists xpy(x = y)$ is true. According to the semantics of L_Q, this is the case if for some $d \in \mathbb{D}$, $I(p) = \{\{d\}\}$. Next, assume that A is a formula in which no constant occurs (A is a classical formula). $A(p\,|\,x)$ is a formula of L which results from A by substituting every free occurrence of x by p. Let I^* be an interpretation of L such that I^* agrees with I on predicate and function symbols and $I^*(p) = d$. Then it is easy to see that: pxA is true in the model $<\mathbb{D}, I>$ iff $A(p\,|\,x)$ is true, in the usual sense, in the model $<\mathbb{D}, I^*>$ (with respect to some fixed variable assignment). This fact can be easily extended to formulae of L_Q containing more constants. Thus, we see that the role of constants in classical logic can be played in **Q** by some 'generalised quantifiers'.

3. A MODAL LOGIC BASED ON N-WORLD SEMANTICS

Let me point to some feature of the possible-world semantics for normal modal logics. As we know, according to the semantics, the formula $\Box A$ is true iff A is true in *all* possible worlds, and the formula $\Diamond A$ is true iff A is true in *some* possible world (here we take into account the simplest version of semantics for **S5**). We can say that in that sense the truth in all, and the truth in some possible worlds is expressible in the language of modal logic. But it is remarkable that the truth in separate possible worlds is not expressible in this way (except the actual one). It might seem that this is because the phrase: "the formula is true in every (some) possible world" and "the formula is true in the possible world w" have different logical structures. However, we may treat the phrases as having the same structure, i.e. treat quantifiers and names as expressions of the same category and – what is especially important here – as categorematic expressions. This idea can be found in R. Montague's work and current semantical analyses of noun phrases by means of the notion of generalised quantifiers.[8] From this point of view only it seems feasible to express the truth in separate possible worlds and in n-worlds as well on the ground of some extended modal language. Notice that our interpretations of n-worlds are just like interpretations of generalised quantifiers on the set of possible worlds. Since the universal and the existential quantifiers are related to the necessity and the possibility operators respectively, the same can be done with other generalised quantifiers. Thus we may adopt **Q** as a logic underlying n-world semantics.

Let $n_1, n_2, \ldots$ be names of n-worlds and let $\Im$ be a function which assigns to the names non-empty subsets of $\mathcal{P}(\mathbb{D}) - \{\emptyset\}$. Now we consider the language L_N which results from the classical modal language by adjoining to it countably many new operators: $\underline{n}_1, \underline{n}_2, \ldots$ The semantic interpretation of operators will be the same as respective n-worlds, whereas $\Box$ will be associated with the universal n-world and $\Diamond$ will be associated with the existential n-world. Given any operator α of L_N, the formula αA will be read as follows: in the n-world associated with α it is the case that A or: A obtains in the world associated with α. In particular, $\Box A$ may be read alternatively: in the universal n-world it is the case that A, and $\Diamond A$ may be read alternatively: in the existential n-world it is the case that A. Let $\hat{\alpha}$ refer to the generalised quantifier

with the same interpretation as α, i.e.: $I(\hat{\alpha}) = \Im(\alpha)$.

The main principle of semantics for L_N will be the following:

(3) A formula αA is true (in a p-world v) iff $\hat{\alpha}x(A$ is true in $x)$ (or: iff there exists a set $X \in \Im(\alpha)$ such that for every p-world $w \in X$, the formula A is true in w).

(Compare the principle with (1) and (2).) The remaining details of the semantics for L_N are the same as those of classical semantics for **S5**. Notice only that, according to (3), the modal operators $\Box$ and $\Diamond$ have their usual meaning.

Needless to say, formulae: αA and $\alpha\neg A$ may be simultaneously true or simultaneously false (i.e. the n-world associated with α is inconsistent or incomplete) and the two formulae together do not entail αB for arbitrary B.

The logic of n-worlds, call it **N**, is axiomatised by the following axioms and rules of inference.

$AN1$ Truth functional tautologies

$AN2$ $\Box(A \supset B) \supset (\alpha A \supset \alpha B)$

$AN3$ $\Box A \supset A$

$AN4$ $\neg\Box\neg\alpha A \supset \Box\alpha A$

MP if $\vdash_N A \supset B$ and $\vdash_N A$ then $\vdash_N B$

NG if $\vdash_N A$ then $\vdash_N \alpha A$ and $\vdash_N \neg\alpha\neg A$

(Compare this axiomatics with the axiomatics of **Q**.) Let us list some characteristic theorems of **Q**.

$TN1$ $\Box A \supset \alpha A$

$TN2$ $\alpha A \supset \Diamond A$

$TN3$ $\alpha(\alpha A \supset A)$

$TN4$ $\alpha\beta A \equiv \beta A$

It is easy to notice that **N** comprises the modal logic **S5**, since the following axioms of **S5**:

$$\Box(A \supset B) \supset (\Box A \supset \Box B)$$

$$\Diamond A \supset \Box\Diamond A$$

are theorems of **N** (cf. $AN2$ and $TN4$). That means that n-world ontology with inconsistent and incomplete worlds does not violate the laws of classical modal logic. As in the case of logic **Q** and classical logic, it is worth emphasising that the completeness of **N** follows in a straightforward way from the completeness of **S5**.

Of course, our interpretation of **N** as a logic of n-worlds is not obligatory. Instead we may treat **N** as a logic of operators (of a certain kind)[9] not mentioning n-world ontology. Let me give just one example of subset of $\mathcal{P}(\mathbb{D}) - \{\emptyset\}$ for which the operator interpretation is much more convincing than the n-world interpretation. The set $\{X \subseteq \mathbb{W} : |X| > |\mathbb{W} - X|\}$ can hardly be interpreted as an n-world but clearly we may associate it with the operator "most often" (cf. so called Rescher's 'plurality quantifier').

N is in a sense the simplest logic with respect to iterations of operators – according to $TN4$ any iteration of operators is reducible to the one immediately preceding the formula. But there are a variety of systems based on language L_N and comprising some normal classical modal system, not necessarily **S5**. Besides, there is a number of systems between **S5** and **N**; especially interesting are these which, instead of $AN4$, contain the weaker axioms: $\alpha\beta A \supset \beta A$ or $\beta A \supset \alpha\beta A$ (consequently $TN4$ is not provable in them). But the semantics of all those systems is more complicated than the semantics of **N** and its development goes beyond the scope of this paper.

4. FURTHER EXTENSIONS

Let us turn back to the logic **Q**. One way of extending **Q** consists in enriching it with operations that enable us to produce new quantifiers from quantifiers already introduced. The extension can be carried out by adjoining to **Q** the following definition axioms:

$DQ1$ $\quad \bar{p}xA \equiv \neg px\neg A$

$DQ2$ $\quad (p \wedge q)xA \equiv pxA \wedge qxA$

$DQ3$ $\quad (p \vee q)xA \equiv pxA \vee qxA$

The new quantifiers receive appropriate semantic interpretations:

$I(\bar{p}) = \{X \subseteq D : X \cap Y \neq \emptyset \text{ for some } Y \in I(p)\}$, $I(p \wedge q) = \{X \cup Y : X \in I(p) \text{ and } Y \in I(q)\}$, $I(p \vee q) = I(p) \cup I(q)$.

The analogical extension can be performed for logic **N**:

$$DN1 \quad \bar{\alpha}A \equiv \neg\alpha\neg A$$

$$DN2 \quad (\alpha \wedge \beta)A \equiv \alpha A \wedge \beta A$$

$$DN3 \quad (\alpha \vee \beta)A \equiv \alpha A \vee \beta A$$

Here, $\bar{\alpha}$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, represent new operators and correlatively, new n-worlds. The semantic interpretation of those operators is defined in the same way as the interpretation of respective quantifiers. $DN1$, $DN2$, $DN3$ express jointly the fact that the ontology of n-worlds is closed under De Morgan lattice operations.

The analogy between **Q** and **N** may be further exploited while taking into account the fact that the interpretation of operators is just like the interpretation of quantifiers on the set $\mathbb{W}$. Let us notice that some logically relevant structural properties of quantifiers are expressed by means of identity – cf. theorems $TQ5$–$TQ10$. Of course, identity is not present in logic N, but it is possible to incorporate it in a certain form. Consider the formula $\hat{\alpha}x\hat{\beta}y(x = y)$ and abbreviate it by $\alpha \blacktriangleright \beta$. Next, assume that any expression of the form $\alpha \blacktriangleright \beta$ is a formula of L_N. According to $Q5$–$Q10$, the following formulae will be theorems of the extended version of **N**[10] (the validity of the theorems can be checked directly by appealing to the semantics):

$$\alpha \blacktriangleright \bar{\alpha} \supset (\alpha A \wedge \alpha B \supset \alpha(A \wedge B))$$

$$\Diamond \blacktriangleright \bar{\alpha} \supset (\alpha\neg A \supset \neg\alpha A)$$

$$\Diamond \blacktriangleright \alpha \supset (\neg\alpha A \supset \alpha\neg A)$$

$$\Diamond \blacktriangleright \bar{\alpha} \wedge \Diamond \blacktriangleright \alpha \supset (\alpha\neg A \equiv \neg\alpha A)$$

$$\alpha \blacktriangleright \bar{\alpha} \supset (\alpha(A \supset B) \supset (\alpha A \supset \alpha B))$$

$$\alpha \blacktriangleright \bar{\beta} \supset (\alpha A \supset \beta A)$$

$$\bar{\alpha} \blacktriangleright \beta \supset (\beta A \supset \alpha A)$$

We see that by means of "$\blacktriangleright$" we are able to characterise closely the logical behaviour of operators and correlatively, the logical structure of n-worlds. In particular, if $\Diamond \blacktriangleright \bar{\alpha}$ then the n-world associated with α is consistent; if $\Diamond \blacktriangleright \alpha$ then the n-world associated with α is complete;

if both $\Diamond \blacktriangleright \bar{\alpha}$ and $\Diamond \blacktriangleright \alpha$ then the n-world associated with α is a p-world; if $\alpha \blacktriangleright \bar{\alpha}$ then the n-world associated with α is closed under Modus Ponens. It should be emphasised that "$\blacktriangleright$" expresses notions even stronger that the notions of completeness and consistency of n-worlds and these are not relativised to any language in which propositions are expressed.

There are many ways of further developing n-world ontology and applying it as a semantic frame to various logics (not necessarily modal ones but, for example, paraconsistent). In particular, in order to extend the logic **N** to predicate logic one would have to know what the internal structure of n-worlds looks like, i.e. what objects inhabit the worlds, what is the nature of predication, identity and quantification. Generally, it seems rather obvious that the internal ontology of n-worlds should be of a Meinongian kind. All details aside, the main idea of such an ontology is that objects are uniquely characterised by classes of properties. An object possesses a property if and only if the property is a member of such class. Since there are no limitations with respect to the size and contents of the classes characterising Meinongian objects, the objects can be incomplete and inconsistent (impossible). Recently there have appeared several formal theories which are either formalisations of Meinong's views or which are inspired by the views[11]. Some of the theories are logics sensu stricto, they are just Meinongian logics. But there are many ways of combining Meinongian ideas with modality and they are quite independent of each other. The way I am suggesting here (from n-worlds to objects of a Meinongian kind) is one of them and is rather unorthodox. Anyhow I would like to stress that any such combination provides an extremely fruitful and fascinating framework for discussing various problems of object existence and identity.

5. SOME APPLICATIONS

When we consider incomplete and inconsistent n-worlds that usually brings to mind fictional worlds which are always incomplete and sometimes inconsistent. That is why the logic **N** may be applied to logical analyses of fiction, i.e. analyses of what is true according to a given piece of fiction and how we reason about what is true. Most often the analyses are carried out by means of a fiction operator. For example, when we take into account 'Sherlock Holmes stories' written by Sir A. C. Doyle the respective operator has the form: 'In the Sherlock Holmes world it

is the case that'. The operator is correlated with the Sherlock Holmes fictional world which can be interpreted as an n-world. Obviously, this n-world is incomplete and perhaps inconsistent.

The process of constructing the fictional operator may be described roughly as follows. If a given fiction is consistent then we assign to the operator the set $\{\{F\}\}$, where F is the set of all p-worlds where the fiction is true. The situation is a slightly more complicated if the fiction is inconsistent. In that case select sets $F_1, F_2, \dots, F_n$ of p-worlds such that for every consistent fragment of fiction there is a set F_i where the fragment is true, and then assign $\{F_1, F_2, \dots, F_n\}$ to the operator. This construction guarantees that for every original sentences A of fiction, A is true in the respective n-world W, i.e. the sentence αA, where α is the fiction operator, is true. But there are more sentences true in W. Notice that according to $N2$ the truth in the n-world is closed under strict implication. This means that everything that is strictly implied by the contents of fiction is true in W as well.[12]

I would like to mention just one more aspect of treating the logic of fiction as a logic of n-worlds based on the language L_N. There is a phenomenon, widely discussed by philosophers, of "fiction in fiction", which occurs if a hero of fiction creates another fiction. In such a case we can distinguish two fictional worlds – the external one created by the real author and the internal one created by a fictional author – and respectively, two fiction operators: α, β. That phenomenon may be formally expressed in L_N by means of iterations of operators. The formula $\alpha\beta A$ will be read: in the external fictional world it is the case that in the internal fictional world it is the case that A. Of course, the logic **N** itself is not adequate for such an interpretation since it trivializes the iterations of operators. We should rather choose in this respect the logic for which there holds only the implication: $\alpha\beta A \supset \beta A$ (i.e. what is true in the internal world according to what is true in the external world, is true in the internal world as well but not conversely).

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NOTES

[1] Cf. [14].

[2] N-worlds are not to be confused with so called "dead ends" and "non-normal worlds" used in

the Kripkean semantics for $S2$ and $S3$. Cf. for example: [4] and [5].

[3] When we say "basically" we mean that this holds if for every two p-worlds there exists a proposition which obtains in one p-world but not in the other.

[4] For the set-theoretical interpretation of the two operations see Section 4.

[5] Rescher and Brandom have not noticed that possibility of enriching the lattice structure of n-world ontology. However, in the author's opinion, the structure cannot be extended to the Boolean structure.

[6] The logic has been developed in details in [11]. From the bulk of current literature concerning generalised quantifiers, their definition and classification, we can recommend for example [1], [16], [17], [18].

[7] The method of proof is similar to that which was for the first time employed by Corcoran, cf. [2].

[8] Cf. for instance [9], [18], [19].

[9] The idea of interpreting operators as sets of sets of possible worlds goes back to Montague, cf. [8] and also [3].

[10] More precisely, we build a mixed logic $N + I$ where I is the fragment of Q which contains no predicate and function symbols except the identity symbol. For any operator α of N and any formula A of I we will stipulate: $\alpha A \equiv A$.

[11] Cf. [6], [10], [12], [13], [15], [20], [21].

[12] Our approach to fictional worlds is very rough. Actually, logical problems of fiction are very complicated. For example see [7].

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AGAINST STRAIGHTFORWARD ANTI-REALISM[1]

The aim of this paper is to examine Michael Dummett's arguments for anti-realism. I will focus mainly on Dummett's reasoning that he claims leads to the abandonment of classical laws of logic in favor of intuitionistic ones. Mathematical intuitionism provides the paradigm for a considerably simple form of anti-realism and, according to Dummett's characterization, rests on abandoning the notion of truth in the account of meaning of logical connectives and quantifiers. It is precisely this abandoning that differentiates the case of intuitionism from most other anti-realistic positions which are based on theses that explain how the meaning of expressions of the disputed language is related to expressions of some other language whose meaning is taken for granted.

As I mentioned before, intuitionism offers a paradigm for a number of similar positions which I call "straightforward anti-realism". I shall characterize these positions later. As was shown by Dummett, the argument for mathematical intuitionism does not rely on any claim about peculiarities of mathematical objects. Hence, if it were possible to show that the argument in favor of intuitionism was substantially wrong, that it involved an appeal to a clearly mistaken assumption and thus, could not be corrected, it would mean that straightforward anti-realism is not a tenable doctrine. It is indeed my contention that, provided Dummett characterization of intuitionism is correct, this sort of anti-realism is indefensible.

Dummett's fundamental idea is that most realist/anti-realist disputes translate into arguments concerning the appropriate semantics and meaning-theory for language people employ in speaking about controversial states of affairs. In order to give some examples of these controversies let me mention debates over reality of future and past, reality of character-traits, reality of mathematical objects. The project of approaching metaphysical questions by establishing first what is the proper account of meaning involves a number of disputable claims. They include: (1) the requirement that meaning-theory should be based on some semantics, (2) equating meaning-theory with a model of understanding,

Jan Woleński (ed.), Philosophical Logic in Poland, 199–211.

(3) the demand of manifestation of linguistic knowledge. However, I will limit myself to the brief exposition of these claims. I will focus on the ultimate argument. This argument claims that the meaning-theories, which choose the concept of truth as their central notion, lead inevitably to ascribing to speakers a sort of knowledge whose nature cannot be accounted for. This knowledge seems to transcend any use a speaker can make out of it. I would argue, contrary to Dummett, that in the framework of such theories, we can say how the knowledge of meaning can be manifested. The answer relies on the idea of compositionality of meaning and it is a surprise that Dummett has never envisaged it. The only reason why the suggested answer is not accessible from Dummett's viewpoint seems to be his acceptance of a particular methodological principle, i.e., every term of a purported meaning-theory should bear a one-to-one correspondence to some specifiable, though interconnected, part of linguistic practice. Because there are no convincing reasons why one should adhere to this idea and a number of reasons why one ought to give it up, I claim that the acceptance of this idea is the mistake of his reasoning. I also believe that his reasoning cannot do without the mentioned mistaken methodological principle. And, if I am right, this dependency undermines the position of straightforward anti-realism.

Let me however first set about a brief exposition of Dummett's project – without it my argument would hardly be intelligible. What is then the distinctive feature of straightforward anti-realism? It can be answered most easily by sketching first, in terms of meaning-theory, the position of a full-fledged realist and then inquiring on what counts it can be abandoned. The salient feature of such a position is a conviction that controversial states of affairs obtain or do not obtain, quite independently whether human beings have or could have the capacity to recognize them as obtaining or otherwise. It means that the realist has at his disposal a concept of truth that makes the principle of bivalence hold for the statements of language employed in the description of the states of affairs in question. Secondly, the realist should adhere to a specific explanation of what renders statements true if they are true. He does this by appeal to classical semantics (model theory), firstly choosing denotation of singular terms as their semantic values, and secondly accepting truth-values as semantic values of sentences. Finally, while explaining the meaning of sentences, the realist invokes the notion of truth-condition: understanding a sentence means knowing what must obtain for the sentence to be true. This idea is not much informative,

and as such, can be agreed on by different parties. What makes the realistic position distinctive is that the concept of truth figuring in the account of meaning coincides with truth as one of the semantic values of sentences.

In most subject-matters, however, the full-fledged realism sketched above is hardly tenable, or at least, it needs much additional support in order to be maintained. Frequently, a language we use to speak about some subject-matter contains seemingly referring expressions, and the question is whether or not such expressions can stand for anything. Such a question divides philosophers into realists and their opponents. One is thus tempted to explain the meaning of sentences involving the "suspicious" vocabulary with the help of some language that does not contain the vocabulary. Traditional examples of this procedure are provided by attempts to explain the class of sentences attributing character-traits with the help of language of behaviorist psychology. In another example, sentences containing terms purporting to refer to material objects are explained in terms of sense-data language. Thus, the meaning of any original sentence is to be explained as the meaning of some (suitably chosen) sentence(s) of the language to which the original one was reduced. This explanation is accomplished either by requiring that there is a translation between the two languages or by demanding that no sentence of original language can be true unless some sentence(s) of the latter language is (are) true. The first, stronger claim is usually called "reductionist thesis", the second, a weaker one, "reductive thesis". Neither claim leads necessarily to the rejection of realism, although realism is endangered by such claims. Briefly speaking, whether realism is rejected here, depends whether the claims allow one to preserve the concept of reference for singular terms of the original language. Hence, the mere adoption of reductive or reductionist theses does not require the original sentences to be interpreted in the way which accords with anti-realism. Whether the sentences can be realistically interpreted depends on a particular scheme of translation or on the details of the reductive thesis.

It thus seems that, if Dummett is right, there are at least three claims inherent in the realistic position concerning some subject-matter: (1) the meaning of a sentence is explained in terms of its truth-conditions, irrespective whether the truth-conditions can be recognized as obtaining, (2) sentences of the discussed class are believed to be subject to the principle of bivalence, and (3) the mechanism explaining how truth-

values of sentences depend on the way a sentence is built out of its constituents is based on the notion of reference. These claims suggest that there are three ways in which one might argue against realism.

The intuitionist is not obliged to claim that mathematical statements receive meaning due to being related to some other statements, possibly not mathematical. One obvious candidate for accounting for the meaning of a mathematical statement, say statement A, is of the form: "There is a proof that A". The application of the reductionist thesis to mathematical statements suggests that the statement claiming the existence of a proof can be understood antecedently to the statement setting forth the content of the theorems. On an extreme formalist account of mathematics, one can claim that the notion of proof is prior to the notion of meaning of a theorem. But in everyday practice, one needs first to understand mathematical statements in order to recognize what a proof of a given theorem is. Besides, the intuitionist hardly agrees with such formalist philosophy of mathematics. On the other hand, the adherent to reductive thesis faces serious difficulties concerning the notion of truth. The reductive thesis applied to mathematical statements has the form: The sentence A is not true unless the sentence "There is a proof that A" is true. What does, however, mean that the sentence claiming the existence of a proof is true? If it means that the proof exist timelessly, independently from human capacities to carry it out, the resulting canons of reasoning will likely be the same as classical. If on the other hand, existence of a proof is understood as somebody's actual possession of the proof, the truth-predicate becomes significantly tensed; that is, there is some sense in claiming that some statement became true on the date such-and-such. For these reasons, the intuitionist position concerning meaning of mathematical sentences is better off when it does not begin with attempts to explain intuitionistic notion of truth, and does not involve either the reductionist thesis or the reductive thesis.

The intuitionistic project should be rather viewed as consisting in accounting for both the meaning of mathematical statements and for the meaning of sentences claiming the existence of relevant proofs. Meaning of mathematical statements is explained in terms of a special sort of proofs called canonical proofs. In Heyting semantics, one stipulates what counts as a canonical proof of any atomic sentence and then defines, for each logical connective and quantifier, what is a canonical proof of a sentence whose main operator is the connective (quantifier) in question. In effect, it is described, for every statement, what its

canonical proof should look like. This allows us to say that the grasp of a sentence consists in the knowledge of what counts as its proof. Accordingly, once a speaker knows what the canonical proofs of subsentences can be and has the grasp of the main operator of the sentence, then he knows the meaning of original sentence. And it is claimed that the understanding of a sentence can be equated with the capacity to recognize something as a canonical proof of a sentence, once it is presented.

Now Dummett's opponent may rise the following questions: (1) Why should considerations concerning meaning of linguistic expressions have any impact on the accepted modes of reasoning, and more specifically, (2) how can some proposed meaning-theory play a role in revising logical laws which are widely observed? Dummett accepts a number of claims that make such a revision possible. The first claim postulates strong relations between a semantics (as a logical theory) and meaning-theory. There are at least three reasons for postulating such a connection:

(1) The meaning-theory should explain what it is to grasp the content of a sentence. It should also allow that knowing the sense of any expression is related to a way of deciding what the expression stands for (or, more generally, what semantic value it has). Semantic theory, on the other hand, selects a notion in terms of which the significance of uttering a sentence is to be characterized, that is what sort of semantic values sentences can have. It asks then how the expressions occurring in a sentence, together with the syntactic form, determine the semantic value of the sentence. Thus, briefly speaking, senses of expressions should elucidate what semantic values the expressions have.

(2) Knowing the content of some sentence must be somehow related to the grasp of what must obtain in order that a sentence be valid or true in the broad sense. This insight is often rendered by saying that to understand a sentence one needs to know its truth-condition. A concept of truth is also employed in semantics: a semantic theory has to show how validity or truth of a sentence depends on semantic values of constituent expressions. Thus, if a meaning-theory employs some notion of truth, this notion should coincide with the concept of truth

as it occurs in a semantic theory.

(3) Finally, a semantic theory validates some set of introduction and elimination rules of inference. Such rules, at least in the case of the language of mathematics, coincide with grounds of asserting a sentence and commitments made by its assertion, respectively. Within meaning-theory, however, there is a demand that the grounds for asserting a sentence should be in a sort of harmony with the commitments made by asserting it. The requirement of harmony, as imprecise as it is, says that the consequences of the assertion should not exceed or be more far-reaching than the grounds on which the assertion has been made. Accordingly, the same requirement applies to introduction and elimination rules; application of an introduction rule for some logical connective followed by application of an elimination rule for the same connective should not allow one to assert a sentence that, on the grounds of assertion of the initial sentence, could not be asserted earlier. Thus, it means that even a well established linguistic practice can be revised when the two features of assertion (in the case of mathematics: introduction and elimination rules) are not in harmony. In order to decide this, however, the notion of harmony should first be made precise.

To complete the picture, let me only mention the last principles involved in Dummett's project.

(1) The meaning-theory is made fallible by equating knowledge of meaning with understanding. The theory should reveal what speakers know when they understand linguistic expressions, sentences, language. This means, however, that the theory can be judged according to whether it accords with some realistic picture of understanding, linguistic communication and language-learning. In particular, a meaning-theory should not ascribe to speakers a sort of knowledge which cannot be shown or manifested.

(2) The last principle, which does not involve much controversy, is the demand of semantical justification of logic. This principle allows one to "close the circle": if a theory of meaning, understood as a model of understanding, has some unaccept-

> able consequences it should be rejected and replaced by a better one. This change may involve revision of the underlying semantics, which in turn may lead to revision of some forms of inference.

We have come to the central point of Dummett's argument for revision of accepted modes of reasoning, which is the accusation that meaning-theory based on classical semantics has insurmountable difficulties in relating the knowledge of meaning to the ways the knowledge can be manifested. Such a theory has problems on two levels: (1) with "basic" expressions and (2) with undecidable statements. I will examine the objections in the following.

Explanation of meaning must break somewhere. It is impossible to state without circularity, for all expressions, what they mean. There are words whose understanding can be judged by a speaker's mastery of their use, but it cannot be answered, at least without circularity, what such words mean. A paradigmatic case for such basic expressions is that of color-words. It is also believed that some fundamental rules (some rules of computation) are of this nature. What is involved in the claim that knowledge of the meaning of a statement in which basic words occur is to be explained as awareness of what should obtain for such a statement to be true? It is not informative to say that somebody understands the statement "These curtains are blue" because he knows what condition must obtain for the statement to be true. Dummett however makes a much stronger accusation, to the effect that, in the truth-conditional theory of meaning, one is forced to concede that the ability of immediate recognition whether a basic expression "applies" to a situation, is explained in terms of ostensive definitions. For example, the ability to recognize that these curtains are blue, results from a speaker's ostensive definition of the word "blue". The rest of Dummett's reasoning follows Wittgenstein's in arguing against ostensive private definitions: there is no relation between "correctness" of such a definition and mastery in using basic expressions. My contention is, however, that an adherent of the truth-conditional theory is by no means obliged to take recourse to ostensive definitions. It is clear that the understanding of these expressions should help a speaker settle what are their semantic-values and how they determine the truth-values of sentences in which they occur. The meaning-theory, however, is silent about what allows a speaker to link basic words with respective qualities, objects etc. It is not a good

answer to say that the expression obtain their meaning by means of ostensive definitions. Instead, one can say that speakers acquire understanding of these expressions by observing how others use them, trying to utter them in various circumstances, being corrected, etc. As the result of these interactions, people get the idea of correct and incorrect utterance in given circumstances. This idea also provides rudiments of the notion of truth. Hence, by saying that somebody understands a sentence (involving some basic expression) because he knows what condition should obtain for the sentence to be true, we claim only that he knows in which circumstances its utterance is correct and in which circumstances it is not. And obviously a listener can judge whether a speaker possesses this sort of knowledge or not.

More perplexing is the second problem, which concerns the sort of knowledge ascribed to somebody who understands an undecidable statement. ("Undecidable statement" here means a statement for which no proof is known.)

The thrust of Dummett's argument amounts to asking what it is to say that understanding of an undecidable statement consists in knowing its truth-condition. There is no known proof for the statement, which means that a speaker has no means to recognize whether the truth-condition obtains or not. But the theory ascribes to him some knowledge (of truth-condition), which should be related to observable behavior. And manifestation of knowledge of truth-condition should mean something more than merely stating the content of the statement in other words.

What is however involved in the demand that knowledge of truth-condition should be capable of being manifested? In the case of decidable statements, it can be explained what the manifestation of the relevant knowledge is. Suppose that in a classroom some student is examined whether or not she understands the sentence: "31 is a prime number". Perhaps the simplest way for her to convince the teacher would consist of observable failed attempts to divide 31 by 2, 3, 4, ..., 30 and then to state that the number is prime. Hence, in Dummett's somehow elaborate terminology, to manifest fully the truth-condition of a sentence, two conditions must be satisfied: (1) the mastery of the relevant decision procedure should be displayed, that is, the procedure which allows one to get into a position where he can recognize the truth-condition as obtaining, if it obtains, and (2) verbal behavior by which an individual acknowledges the condition as obtaining if it obtains (or as failing to obtain if it fails) should be observed. It is important to

notice that, in order for the knowledge of truth condition were fully manifested, both clauses should be satisfied.

However, why does Dummett insist that nothing less than mastery of relevant decision procedure can serve as evidence for knowing the truth-condition of an undecidable statement? Let us suppose that, for some reason, the mastery of a decision-procedure for a statement cannot be displayed. Assume however that we have strong evidence that all the expressions occurring in the sentence are understood by a speaker in the way we understand them. Suppose further that we have obtained this evidence from observing how these expressions are handled when they occur in decidable statements; that is, we have seen what their impact is on decision-procedures for sentences containing them. Because knowledge of truth-conditions of decidable statements is related simply to decision-procedures, it can be said that our observations give us information on how each of the expressions contribute to the determination of truth-conditions of decidable statements.

One can imagine the following discussion between the adherent (A) and the opponent (O) of truth-conditional theory:

(A) I acknowledge that I do not know any simple explanation of what the grasp of the truth-condition of an undecidable statement is. I suggest that you treat the term "knowledge of the truth-condition" as a sort of theoretical term within the theory of meaning, and you need not bother much what the term "knowledge of the truth condition of an undecidable statement" stands for, provided that the theory allows you to give an overall picture of meaning.

(O) Yes, but the theory leads to troublesome consequences. Suppose two individuals are engaged in proving an undecidable statement, say Goldbach's conjecture. Let's say they have proved several lemmas relating the statement to other theorems, tested several alleged proofs, launched even a computer program for finding some even number which is not the sum of two primes. Such undertakings however remained unsuccessful in deciding whether the conjecture holds. Now does the theory allows you to ascribe to them the same understanding of the statement? All displayed abilities fall short of being a full manifestation of the truth-condition. You cannot even tell whether displayed abilities manifest partially the knowledge of the truth-condition of the sentence "The conjecture is indetectably true", or manifest the knowledge of the truth-condition of the statement "The conjecture is indetectably false". But definitely, there is some difference in meaning of the above sen-

tences. Therefore, how can your theory grant that people communicate linguistically?

(A) Hold on, please. You certainly agree that in the case of sentences for which decision-procedures are known, you can safely talk about knowledge of their truth-conditions. You can even construct meaning of a constituent expression from the way it determines truth-conditions of the sentences in which it occurs. Now, because a theory of meaning mirrors semantics, meaning of a sentence should depend on meanings of its constituents. So, in the case of an undecidable statement, you know its truth-condition provided you know the "contribution" to determining truth-conditions that each constituent makes. And the way an expression determines truth-value of such a statement can be, in principle, known from observing how the decidable statements are handled.

(O) What is the guarantee that some expression figuring in an undecidable statement contributes to the statement's meaning in the same way that the syntactically same expression contributes to the meaning of a decidable one? In your model, you cannot verify that somebody working on, say, Goldbach's conjecture, attaches the same meaning to both the conjecture's universal quantifier and the universal quantifier occurring in a decidable statement.

(A) I do not think that such a guarantee is necessary. It is quite natural to assume that meaning of any expression does not depend on whether it occurs in a decidable or an undecidable statement. Whoever is ready to seriously entertain this possibility must be prepared to acknowledge that meaning changes as a statement is being decided.

The idea that understanding of a statement can be shown, in addition to the manifestation of the decision-procedure, by the ability to use the statement's constituent expressions may be viewed as the extension of the requirement that meaning must be compositional, that is, the content of a complex expression should be derived from its composition and from meanings of the words that build the expression. The only thing that would make this idea unsound is the assumption that meaning of some expressions hinges heavily on whether they figure in decidable or undecidable statements. This assumption, however, is highly unreasonable.

What is more, the number of linguistic devices that bring "undecidability" into language is limited, and, in the case of the language of arithmetics, these devices are quantifiers. Obviously, some statements remain decidable although their principal operator is a quantifier. Hence,

by observing how a speaker recognizes whether the truth-condition of a decidable statement with a quantifier as the main operator obtains, it is possible to judge whether he understands the quantifier in question. If the quantifier's meaning does not change, the mentioned behavior counts as partial evidence for his understanding of a respective undecidable statement.

It is however very surprising that Dummett, for whom the principle of compositionality of meaning is so dear, never envisages the possibility that knowledge of truth-condition of a sentence can be construed as a grasp of meaning of its constituents (again explained in terms of their impact on truth-conditions of other sentences), and that full manifestation of the knowledge of the truth-condition may also consist of the display of understanding of its constituting expressions (i.e., in showing mastery in deciding statements in which the relevant expressions figure). Perhaps Dummett fears that this position leads to a kind of "holism", that "no sentence of a language can be fully understood unless the entire language is understood." (1977 p. 366) The idea that some sentence cannot be understood until some others are grasped accords well with our everyday practice. What holism rejects however, is an ordering of sentences with respect to their "complexity" such that, in order to know the meaning of some sentence, it is not necessary to understand sentences of greater "complexity". Clearly, while investigating whether a speaker understands the universal quantifier, one can choose any decidable statement and observe how the speaker carries out the relevant decision-procedure. Hence, one cannot impose any limitation on the complexity of decidable statements which can be investigated to give information about how the speaker understands the quantifier.

However, similar danger threatens meaning-theory which accords with intuitionism. For example, somebody understands an implication-sentence if he knows that the proof of this sentence should be an operation which transforms the canonical proof of the antecedent into the canonical proof of the consequent. In the operation any sentence of arbitrary complexity can occur, hence, to avoid the holistic conclusion, Dummett concedes that, for understanding of such a sentence, it is not necessary to know the details of the operation, and that grasp of the notion of effective procedure suffices. (Dummett 1991, p. 227) This concession has disastrous consequences. If somebody understands some implication-sentence only because he is acquainted with the notion of effective procedure and not because he has the ability to understand

all the sentences stating the effective procedure, then he is not capable of recognizing the canonical proof of that sentence if such a proof is presented to him. The main advantage of "intuitionistic" meaning-theory, namely equating the understanding of a sentence with abilities to recognize its proof, is gone. Hence, Dummett's intuitionist faces a dilemma: either he surrenders to holism, or he gives up his position's main advantage over mathematical realism.

It is thus apparent that it is not holism that forces Dummett to demand that the only manifestation of knowledge of the truth-condition of a sentence is mastery of the relevant decision-procedure.

What does force Dummett is some methodological principle he embraces that can hardly get along with the approach sketched above. He seems to demand that each term of a theory of meaning corresponds one-to-one to some specifiable, distinguishable sort of abilities constituting understanding of a language (1977 p. 377). This demand applies also to the crucial term: "knowledge of truth-condition". The account of meaning of undecidable statements which is advocated here, seems to be in conflict with the methodological demand. Simply, if we grant that a speaker understands an undecidable statement, it is difficult to say what other sentences should he understand and what decision-procedures should he be capable of carrying out.

But why should we adhere to Dummett's methodological principle? As it stands, the principle is hardly observed in natural sciences. It does not seem possible to relate any term of scientific theory to some unique range of possible observable phenomena. Thus, Dummett's demand, as explained above, is merely the methodological principle of the theory of meaning. The principle comes from assuming that a theory of meaning should not be compared, say, to physics; it should neither be a science that postulates posits nor a science having theoretical vocabulary. Instead, it ought to account, in common-sense terms, for what it is to understand a language.

It turns out that the argument for a sort of anti-realism for which intuitionism provides the paradigm hinges heavily upon some methodological principle, which taken generally, is, if not false, then at least highly dubious. In addition, justification of the principle invokes the claim of specific character of theory of meaning. However, there is presently no accomplished theory of this kind, and one cannot insure that there will be any in the future. It means that Dummett's entire argument relies on a principle of a non-existing theory.

What morals, if any, may be drawn from the failure of the argument for mathematical intuitionism, or more precisely, from the fact that it relies on some methodological principle of a non-existing theory? Obviously the failure neither shows that intuitionism is untenable nor that the classical way of practicing mathematics is the only correct one. Also, by no means does it prove that anti-realism is a false doctrine (or, rather a false bunch of doctrines). I claim, however, that a straightforward sort of anti-realism for which intuitionism is a paradigmatic case can hardly score a victory with its opponent. Simply, on that path there are no means of discarding the verification-transcendent notion of truth, unless you count on solving some speculative methodological problem.

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NOTE

[1] I presented the first version of this paper at the Boston Colloquium in Philosophy of Science on January 28, 1992. I am indebted to the participants of the Colloquium for the discussions on this topic. I owe also my gratitude to my colleagues Paweł Turnau and Tom Tonon, who have read the paper and suggested a number of corrections

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A MINIMAL IMPLICATIONAL LOGIC

The system L_c of purely implicational logic is constructed. It is weaker than the Hilbert's implicational logic L_H, but the deduction theorem and some kind of modus ponens rule are valid for L_c. Hence L_c can be considered as a system which characterizes the standard implication connective.

1.

In the opinion of many logicians a propositional system L determines the implication connective in the proper way, if at least two following conditions are satisfied:

(a) The modus ponens rule

(MP) $\alpha \to \beta, \alpha/\beta$

is a derivable rule in L,

(b) the deduction theorem

(D) $\beta \in Cn(X \cup \{\alpha\}) \Rightarrow (\alpha \to \beta) \in Cn(X)$

holds for the consequence operation Cn connected with L.

Of course, α, β are formulae, X is a set of formulae of a fixed language S: all propositional variables and all implications built up from these variables belong to S. Elements of S will be called well-formed formulae or in short wff's.

The conditions (a) and (b) hold for Hilbert's purely implicational, propositional logic L_H which is simultaneously the weakest implicational logic determining the notion of standard implication (cf. [3]).

To this assumed point of view we would like to add a remark. Namely, if we agree that in the purely implicational logic the modus ponens rule can conclude only implicational formulae, then we are able to characterize the implication connective by a logic weaker than L_H, obtaining nearly the same characterization of the implication connective.

Jan Woleński (ed.), Philosophical Logic in Poland, 213–216.

2.

Now we introduce a formal axiomatic theory L_c. The symbols of L_c are $\rightarrow, (,)$ and the letters p_k with positive integers k as subscripts. All statement letters p_k are well-formed formulae and if α, β are well-formed formulae, so is $(\alpha \rightarrow \beta)$. An expression is a well-formed formula (wff) only if it can be shown to be a well-formed formula on the basis of the above clauses. (We can omit the parentheses if it does not lead to misunderstanding).

If α, β, γ are any wff's of L_c, then the following are axioms of L_c

$$\alpha \rightarrow \alpha \tag{2.1}$$

$$(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma)) \tag{2.2}$$

$$(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)) \tag{2.3}$$

The infinite set of axioms of L_c is given by means of three axiom schemas (2.1)–(2.3), each schema standing for an infinite number of axioms. The set of all axioms is denoted by A_c.

The rules of inference of L_c are rules given by the following schemas

$$\beta / \alpha \rightarrow \beta \tag{2.4}$$

$$\alpha \rightarrow (\beta \rightarrow \gamma) / \beta \rightarrow (\alpha \rightarrow \gamma) \tag{2.5}$$

$$\alpha \rightarrow \beta, \alpha \rightarrow (\beta \rightarrow \gamma) / \alpha \rightarrow \gamma \tag{2.6}$$

The set of these rules is denoted by R_c. The axioms A_c and primitive inference rules R_c compose the axiomatic system L_c.

Note, that on the grounds of (MP), the axioms A_c are weaker than the axioms of Hilbert's logic L_H, so for the system composed of A_c and (MP) the deduction theorem is not valid.

The logic L_c is weaker than L_H: the modus ponens rule (MP) is not derivable in L_c, although it is permissible in L_c and the rule of the scheme $\alpha, \alpha \rightarrow (\beta \rightarrow \gamma) / \beta \rightarrow \gamma$, which is a subrule of the modus ponens rule (MP), is derivable in L_c. The deduction theorem (D) is valid for the consequence operation generated by L_c. Moreover the set of theorems of the logic L_c and the set of theorems of Hilbert's implicational logic L_H are the same. But, of course, the sets of derivable rules of these logics are different – it is known that L_H is structurally complete (cf. [2]), whereas L_c is structurally incomplete and its strengthening by the rule of substitution for propositional variables is also structurally incom-

plete. However, L_c is saturated (in the sense defined by Wójcicki, cf. [4]) so one can say that L_c has a standard, regular internal construction.

All these properties point out rather clearly that L_c, the system essentially weaker than L_H, gives a nearly standard characterization of the implication connective.

3.

Some of the properties of the system L_c can be generalized. Let M_2 be the classical, purely implicational matrix (of the language S), let $Sb(X)$ be the set of all substitutions of the expressions belonging to X. The logic L_c may be considered as the pair $(R_c, A_c) = L_c$ and for any $A = Sb(A) \subseteq E(M_2)$ the pair $(R_c, A_c \cup A)$ will be called the axiomatic extension of L_c. So in every axiomatic extension of L_c the (MP) rule is not derivable but is permissible; for every such extension of L_c the deduction theorem holds. Let S' denote the purely implicational subset of the set S. Cn_H, Cn_c denote the consequence operations connected with the logic L_H, L_c respectively. Then we have $Cn_H(X) \cap S' = Cn_c(X) \cap S'$, for any set $X \subseteq S$. Every axiomatic extension of L_c is structurally incomplete but it is saturated.

4.

We say that (D) holds for the logic L or that L has the property (D), if (D) holds for the consequence operation generated by L. It is known that there exists a minimal implicational logic with the property (D) (cf. [1]). This logic is composed of an infinite set of axioms and a single primitive rule of inference. Of course, the existence of such a logic does not exclude questions on logics which are founded on another fixed set of rules of inference and are minimal with respect to (D). So, for example, Hilbert's implicational logic L_H is the (MP)-minimal logic with respect to (D). Similarly, the logic L_c is the R_c-minimal logic for which the deduction theorem (D) holds. It means that the set of axioms A_c is, on grounds of R_c, the weakest set of axioms for which the deduction theorem can be proved.

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KNOWLEDGE REPRESENTATION SYSTEMS FOR GROUPS OF AGENTS

1. INTRODUCTION

Investigations concerning a systematic approach to reasoning by one or more fully communicating, intelligent, agents have been developed by both logicians and computer scientists. In the last few years a number of papers addressing the problem of reasoning about knowledge have appeared, e.g. [H], [Hi], [L], [O], [P], [R] and [RM]. There are, however, many approaches in this area, as for example, how to understand, represent and manipulate knowledge, e.g. [FHV], [Hi], [Ho].

The concept of knowledge presented in this paper is based on the rough set approach [P1], [P2]. The main issue we are interested in is reasoning from imprecise data.

In the paper we assume that there is a collection of agents $t_1, \ldots, t_n$ who observe the same universe of discourse and can communicate among themselves. The basic notions introduced in Section 2 do not depend on whether they are defined for one agent or for a group of agents. Therefore, by t we denote in the paper an agent or a group of agents, unless it is stated otherwise.

We will assume that every agent possesses an image or knowledge about the universe of discourse, shortly the universe. The perception of the universe U by an agent t depends on her abilities, her access to certain tools which she may use in her description on U. In other words, her knowledge about the universe U may be understood as potential knowledge about characteristic features of U. This knowledge is accquired, for instance, from parents, from school or it is a result of her own experiences. It can be extended by the agents perception of some phenomena. If an agent wants to describe a real world she tries to recognize real objects by comparing their properties with those, which according to her knowledge, objects should have. In fact, it is not so easy to define precisely characteristic features of some concepts or entities.

Jan Woleński (ed.), Philosophical Logic in Poland, 217–238.

The lack of precision may follow from incomplete information or the concept by itself may be understood not precisely. As an example take, for instance, the concepts of 'beautiful girl' or 'fat man'. These two notions are examples of a vague concept. Their vagueness is manifested by the existence of examples of these concepts in the border region, that is, the existance of the set which does not contain e.g. any definitely (for an agent t) fat man and any definitely (for an agent t) 'bag of bones' but which contains a man the agent t is not able to decide about his posture. Hence, the vagueness of these concepts makes it impossible to define in a unique way the set of all "beautiful girls" or the set of all 'fat people'.

However, the notion of 'vague concept' seems to be also imprecise. For example, imagine the following situation: We ask a young child to find in a dark room, full of furniture, a small round table. The child has in his mind an image of the required table, or we define what we want her to find as a furniture with a horizontal surface leaned on one or more (but not more than four) legs. The child found a thing satisfying the above description. However, it turned out that it was a bar stool. The concept of 'small round table' seems to be a quite precise one. On the other hand, it is not so easy to define distinction between 'round table' and 'bar stool'.

Our approach to that problem will be based on the following assumptions: Knowledge of an agent about the universe may be complete or incomplete. Abilities of different agents may be comparable or non-comparable. Knowledge of agents about the reality is reflected by their abilities to recognize objects which are positive examples of that reality and objects which are negative examples of it. For instance, if an agent *knows blue objects* then she is able to point out for any object whether it is green or not.

In the paper we investigate formal systems with epistemic operators: partial knowledge operators and knowledge operators. The partial knowledge operator of an agent t is denoted by I_t. The interpretation of a formula of the form $I_t\phi$ is as follows: $I_t\phi$ is the set of all objects from the universe which agent t recognizes as objects having properties given by the formula ϕ. A set of objects X is definable by an agent t if there is a formula ϕ such that the interpretation of $I_t\phi$ is X. If for X such a formula does not exist then X is said to be a rough set. Syntactically, partial knowledge operators correspond to necessity operators in modal logic. Thus, semantically a partial knowledge operator may be interpreted as an interior operator. Hence, the pair (U, I_t) is a topological

space, where U is the universe of discourse. If X is a subset of U, then by the above assumptions the knowledge of t about X, denoted by K_tX, is the union of two sets: I_tX and $I_t(-X)$, where I_tY is the set of all objects which agent t recognizes without any doubt as objects from Y. The knowledge of t about X is complete if $K_tX = U$, that is, the borderline region of X is empty.

Logics to be considered in this paper are extensions of decision logic discussed in [P3] and [Ra] by adopting in the language of decision logic partial knowledge operators. Semantics of these logics is based on rough set theory (cf. [P2]).

We provide complete axiomatization of two formal systems. The first system, t- logic describes reasoning about knowledge of one agent (or a group of agents) and the latter T-logic describes reasoning about knowledge of groups of fully communicating agents.

In both systems the knowledge operator K_t is not used as a primary logical connective. It is defined by means of partial knowledge operators. The advantage of this approach is an intuitive and simple axiomatization.

Finally, let us mention that the knowledge operator K_t is nonmomotonic. As a consequence, logics discussed in this paper avoid well known paradoxes [Hi] of epistemic logic as 'what an agent knows is true' and 'an agent always knows all the consequences of her knowledge'.

The problem of a complete axiomatization of logic for groups of agents was announced in [O], see also [V]. Let us point out that in [R] and [RM] poset-based approximation logics connected with Post-algebras are described as logics of groups of intelligent agents. However, our approach differs from presented in the above papers.

2. PRELIMINARIES

2.1. *Knowledge representation systems*

In this section we present basic notions related to a knowledge representation system of an agent t on the universe U.

In our approach each agent perceives the same reality, the same universe of discourse represented as a set of objects. By objects we mean anything a human being can think of, i.e. real things as chairs, animals, etc., states, abstract concepts, processes. Every agent perceives objects

by means of their features or properties called attributes. Perception of the universe U by an agent t depends on a set of attributes A_t. In other words, her knowledge about U is reflected by her abilities. In a different situation, that is, if the set of attributes changed then her perception of the world may also has changed.

A *knowledge representation system over the universe of discourse U of an agent t,* shortly $k.r.s.$ is a pair $\mathcal{S}_t = (U, A_t)$, where

U – a nonempty, finite set of objects, called the *universe of discourse*,

A_t – a finite set of *attributes* i.e. $a : U \longrightarrow V_a$ for $a \in A_t$, where V_a is called *the value set of a.*

The set $V_t = \bigcup_{a \in A_t} V_a$ is said to be a domain of A_t.

Notice that $k.r.s.$ $\mathcal{S}_t$ is defined for one agent t. However, t is treated in the definition of $\mathcal{S}_t$ as an index only. Hence, t may be considered as a group of agents. From now on in all definitions in this section t may be understood also as a group of agents. Moreover, we will assume that different agents perceive the same universe of discourse.

For any $\mathcal{S}_t$ and any object o from U, $o \mapsto (a, v)$ means that the agent t perceives o as (a, v). For instance, if an attribute a is *color*, its value v is *green* then $o \mapsto (color,\ green)$ means that for the agent t, object o is green.

Instead of $o \mapsto (a, v)$ we will also write $a(o) = v$.

With every subset of attributes $A \subseteq A_t$ we associate a binary relation $ind(A)$, called *indiscernibility relation*, and defined as follows:

$$ind(A) = \{(x, y) \in U \times U : \forall a \in A\ a(x) = a(y)\}\ .$$

Then $ind(A)$ is an equivalence relation and

$$ind(A) = \bigcap_{a \in A} ind(a)\ ,$$

where $ind(a)$ means $ind(\{a\})$. Moreover, for any sets of attributes A and B,

$$\text{if } A \subseteq B \text{ then } ind(B) \subseteq ind(A)$$

and

$$ind(A \cup B) = ind(A) \cap ind(B).$$

Notice that objects x, y satisfying the relation $ind(A)$ are indis-

cernible with respect to the attributes from A. In other words, an agent t cannot distinguish x from y in terms of attributes in A. By $[o]_{A_t}$ we denote the equivalence class of $ind(A)$ including the object o, i.e. the set $\{y \in U : o\ ind(A)\ y\}$. Clearly, for every $A \subseteq A_t$ the family of all equivalence classes of the relation $ind(A)$ is a partition of U.
The partition determined by the set of all attributes A_t, that is, the family

$$\mathcal{E}_t = \{[o_1]_{A_t}, \ldots, [o_n]_{A_t}\}$$

is said to be the *knowledge base of an agent t* over the universe U determined by $\mathcal{S}_t$. Let us emphasize that the knowledge base of an agent depends on the possibility of the perception of the universe U, that is, it depends on a set of attributes A_t.

Example 1
Suppose that the universe of discourse $U = \{o_1, o_2, o_3, o_4, o_5\}$ is given. Let an agent t perceive U by means of the set of attributes $A_t = \{a, b, c\}$ and let $V_a = V_c = \{0, 1\}$ and $V_b = \{0, 1, 2\}$. Suppose that her knowledge representation system over U is given by the table:

U	a	b	c
o_1	1	0	1
o_2	1	0	1
o_3	1	2	0
o_4	0	1	0
o_5	0	1	0

It is not difficult to observe that in that case the relation $ind(A_t)$ establishes the following partition of U: $\mathcal{E}_t = \{[o_1]_{A_t}, [o_3]_{A_t}, [o_4]_{A_t}\}$, where $[o_1]_{A_t} = \{o_1, o_2\}$, $[o_3]_{A_t} = \{o_3\}$ and $[o_4]_{A_t} = \{o_4, o_5\}$. In other words, the partition $\mathcal{E}_t$ is the knowledge base of t about U provided t uses the attributes in A_t to perceive U.

Example 2
Another example of $k.r.s$ may be obtained if a set of agents is treated as a set of attributes. For instance, let $t_1, \ldots, t_n$ be judges in the '92 World Figure Skating Championship. For every $i \leq n$, t_i evaluates the same competitor at the same performance. Every judge t_i creates a record for a competitor x (this is a row in her $k.r.s.$ $\mathcal{S}_t$) and the final score given by him to x results from values for particular fields in the

record. The positition of a competitor x in the competition depends on the common evaluation of all judges $t_1, \ldots, t_n$. So, $k.r.s.$ $\mathcal{S}$ of $T = \{t_1, \ldots, t_n\}$ is a pair (U, T), where U is a set of all competitors. For every $t \in T$, $t(x) = v$ means that the judge t gave to x the note v.

Notice that as a value v in $\mathcal{S}$ one can take the whole record, that is the whole description of x in $\mathcal{S}_t$.

Let U be an arbitrary but fixed universe of discourse. Suppose that $\mathcal{E} = \{\mathcal{E}_t\}_{t \in T}$ is the family of all partitions of U and consider the set of indices T as a set of agents. Notice that $\mathcal{E}_s$ is a partition of U, then there is a knowledge representation system $\mathcal{S}_s = (U, A_s)$ such that $\mathcal{E}_s$ is the partition determined by $ind(A_s)$. Hence, from now on, we assume that any partition of U is treated as a knowledge base of an agent.

We will assume that if for some $t, s \in T$, $A_t \cap A_S \neq \emptyset$ then for every $a \in A_t \cap A_s$, agents t and s perceive every object in the same way, that is, as (a, v), $v \in V_a$. If t perceives an object o as *green* and s as *yellow* then there is no consensus about the object o. In that case there is no knowledge representation system of $\{t, s\}$ having color as an attribute.

For every $s, t \in T$ we are going now to define a join knowledge base of s and t and a common knowledge of s and t. Intuitively speaking, a join knowledge base of s and t is a partition of the universe U such that better recognition of a set of objects by one agent is preserved. Better knowledge about objects is accepted by the group st. For instance, let the knowledge base of s and t about the universe $U = \{o_1, o_2, o_3, o_4, o_5\}$ be the following: $\mathcal{E}_s = \{\{o_1, o_2\}, \{o_3\}, \{o_4\}, \{o_5\}\}$ and $\mathcal{E}_t = \{\{o_1, o_2\}, \{o_4\}, \{o_3, o_5\}\}$. Knowledge of s about object o_3 is more efficient than knowledge of the agent t. So $\{o_3\} \in \mathcal{E}_{st}$. Also $\{o_4\} \in \mathcal{E}_{st}$. Knowledge of t and s about the set $\{o_1, o_2\}$ is the same, hence this set belongs to a join knowledge base of s and t, denoted by $\mathcal{E}_{st}$. Thus we have $\mathcal{E}_{st} = \{\{o_1, o_2\}, \{o_3\}, \{o_4\}, \{o_5\}\}$.

A common knowledge base of s and t is a partition of U such that a weaker knowledge about objects dominates in the group st. In that case only these blocks which occur in $\mathcal{E}_s$ and $\mathcal{E}_t$ occur also in a common knowledge base of s and t, denoted by $\mathcal{E}_{sot}$. For instance, let the universe U be as before and let $\mathcal{E}_s = \{\{o_1, o_2, o_3\}, \{o_4\}, \{o_5\}\}$ and $\mathcal{E}_t = \{\{o_1, o_2\}, \{o_3, o_4\}, \{o_5\}\}$. Because of the agent t does not distinguish o_3 from o_4 then the group of agents st also does not distinguish them and moreover because of the agent s identifies objects o_1 and o_2 with o_3 then $\{o_1, o_2, o_3, o_4\} \in \mathcal{E}_{sot}$. The block $\{o_5\}$ belongs to $\mathcal{E}_s$ and $\mathcal{E}_t$. So

$\mathcal{E}_{sot} = \{\{o_1, o_2, o_3, o_3, o_4\}, \{o_5\}\}$.

To describe precisely definitions of the above notions let for every $s, t \in T$ the relation $\prec$ be defined on $\mathcal{E}$ as follows:

$$\mathcal{E}_s \prec \mathcal{E}_t \text{ if and only if } \forall [x]_s \exists [y]_t \; [x]_s \subseteq [y]_t.$$

It is not difficult to observe that $(\mathcal{E}, \prec)$ is a poset. Moreover, for every $\mathcal{E}_s$ and $\mathcal{E}_t$ the *infimum* of $\mathcal{E}_s$ and $\mathcal{E}_t$ exists in $(\mathcal{E}, \prec)$. Indeed, the set

$$\{[x] \; : [x] = [x]_s \cap [x]_t\}.$$

is a partition of U and it is equal to $inf\{\mathcal{E}_s, \mathcal{E}_t\}$, which will be denoted by $\mathcal{E}_s \cap \mathcal{E}_t$.

Notice that the set $\mathcal{E}_s \cap \mathcal{E}_t$ is a partition of U determined by the relation $ind(A_s \cup A_t)$, where $A_s \cup A_t$ is the set of all attributes in the system which may be viewed as a concatenation of two tables representing $\mathcal{S}_s$ and $\mathcal{S}_t$.

Now, let $\mathcal{E}_s$ and $\mathcal{E}_t$ belong to $\mathcal{E}$. Notice that the set

$$\{[x] : [x] = \bigcup_{[z]_s \cap [y]_t \neq \emptyset} [z]_s \cup [y]_t\}$$

is a partition of U and, moreover, it is the *supremum* of $\mathcal{E}_s$ and $\mathcal{E}_t$ in $(\mathcal{E}, \prec)$. Denote $sup\{\mathcal{E}_s, \mathcal{E}_t\}$ by $\mathcal{E}_s \circ \mathcal{E}_t$.

As a consequence we conclude:

Lemma 2.1 $(\mathcal{E}, \circ, \cap)$ *is a lattice.*

Let the relation $\leq$ be defined on the set of all agents T in the following way: for every s and t,

$$s \leq t \text{ if and only if } \mathcal{E}_t \prec \mathcal{E}_s.$$

The intuitive meaning of the relation $\leq$ is, that, if $s \leq t$ then the sharpness of perception of U and therefore feature recognitions of objects from U by agent s is weaker than that of agents t. For instance, let $\mathcal{E}_s = \{\{o_1\}, \{o_2, o_3, o_4\}, \{o_5\}\}$ and $\mathcal{E}_t = \{\{o_1\}, \{o_2, o_3\}, \{o_4\}, \{o_5\}\}$. Then we have that $s \leq t$, that is, the classification of objects from the universe U by the agent t is better than the classification of U by s. If $s \leq t$ than we say that agent s is less efficient or weaker than agent t.

Clearly, the relation $\leq$ is reflexive, antisymmetric and transitive, that is, the relation $\leq$ is a partially ordering relation on T. Thus, one can consider T as a poset $(T, \leq)$.

It is easy to prove that $inf\{s, t\}$, denoted by $s \circ t$, and $sup\{s, t\}$, denoted by $s \cup t$ or st of s and t exist in $(T, \leq)$. Namely, we have

$$z = s \circ t \text{ if and only if } \mathcal{E}_z = \mathcal{E}_s \circ \mathcal{E}_t$$

and

$$z = s \cup t \text{ if and only if } \mathcal{E}_z = \mathcal{E}_s \cap \mathcal{E}_t.$$

As a conlusion we have

Lemma 2.2 $(T, \cup, \circ)$ *is a lattice with the zero and unit element.*

One can show that the lattice $(T, \cup, \circ)$ is not distributive one and we call it an *agent lattice.*

Now, by a *join knowledge base* of s and t, that is, a *join knowledge base* of the group of agents st we mean the partition $\mathcal{E}_s \cap \mathcal{E}_t$ denoted also $\mathcal{E}_{st}$. By a *common knowledge base* of the group of agents $s \circ t$ we mean a partition $\mathcal{E}_s \circ \mathcal{E}_t$ denoted also $\mathcal{E}_{s \circ t}$.

Now, let **0** denote the zero element and let **1** denote the unit element of the agent lattice $\mathcal{T}$. For any $t, s \in T$ put

$$t \Rightarrow s = \begin{cases} 1 & \text{if } t \leq s \\ s & \text{otherwise} \end{cases}$$

By a T-algebra we mean the structure $\mathcal{T} = (T, \cup, \circ, \Rightarrow, \mathbf{0}, \mathbf{1})$, where the operations $\cup, \circ, \Rightarrow$ are defined above.

Theorem 2.3 *Let $\mathcal{T}$ be a T-algebra. The reduct $(T, \cup, \circ, \mathbf{0}, \mathbf{1})$ of $\mathcal{T}$ is a lattice with the zero element and the unit element and the reduct $(T, \Rightarrow, \mathbf{1})$ of $\mathcal{T}$ is an implicate algebra.*

Example 3

Let three agents $s,\ t,\ w$ perceive $U = \{o_1, o_2, o_3, o_4, o_5\}$ in the following way:

$\mathcal{S}_t$:

U	a	b	c
o_1	1	2	3
o_2	0	1	3
o_3	0	1	3
o_4	2	1	2
o_5	1	3	2

$\mathcal{S}_s$:

U	c	d	e
o_1	3	0	2
o_2	3	0	2
o_3	3	1	0
o_4	2	4	1
o_5	2	4	1

$\mathcal{S}_w$:

U	f	g
o_1	1	1
o_2	0	0
o_3	0	0
o_4	0	0
o_5	2	2

Observe that their knowledge bases are as follows: $\mathcal{E}_s = \{\{o_1, o_2\}, \{o_3\}, \{o_4, o_5\}\}$, $\mathcal{E}_t = \{\{o_1\}, \{o_2, o_3\}, \{o_4\}, \{o_5\}\}$, and $\mathcal{E}_w = \{\{o_1\}, \{o_2, o_3, o_4\}, \{o_5\}\}$.

Then the join knowledge representation system $\mathcal{S}_T$ of t, s and w may be depicted as follows:

$\mathcal{S}_T$:

U	a	b	c	d	e	f	g
o_1	1	2	3	0	2	1	1
o_2	0	1	3	0	2	0	0
o_3	0	1	3	1	0	0	0
o_4	2	1	2	4	1	0	0
o_5	1	3	2	4	1	2	2

The knowledge base of T is the partition $\mathcal{E}_T = \{\{o_1\}, \{o_2\}, \{o_3\}, \{o_4\}, \{o_5\}\}$.

The common knowledge representation system of the agents t and s may be represented by the table given below:

$\mathcal{S}_{sot}$:

U	c
o_1	3
o_2	3
o_3	3
o_4	2
o_5	2

The common knowledge base of t and s consists of two classes, namely, $\mathcal{E}_{sot} = \{[o_1], [o_4]\}$.

2.2. *Knowledge operators*

In this section we want to define a fundamental concept of this paper. We recall first some basic topological notions.

A *topological space* is a pair $(X, \mathcal{D})$ consisting of a set X and a family $\mathcal{D}$ of its subsets which is subjected to the following conditions:

1. $\emptyset \in \mathcal{D}$, $X \in \mathcal{D}$.
2. If $Y_1 \in \mathcal{D}$ and $Y_2 \in \mathcal{D}$, then $Y_1 \cap Y_2 \in \mathcal{D}$.
3. If $Y_i \in \mathcal{D}$ for every $i \in I$, where I is an arbitrary set, then $\bigcup_{i \in I} Y_i \in \mathcal{D}$.

For instance, take the knowledge representation system $\mathcal{S}_t = (U, A_t)$ of an agent t. Then the pair $(U, \mathcal{B}_t)$ is an example of a topological space, where $\mathcal{B}_t = \mathcal{E}_t \cup \{\emptyset\} \cup \{U\}$ and $\mathcal{E}_t$ is the knowledge base of the agent t.

The family $\mathcal{D}$ is called a *topology* and subsets belonging to the family $\mathcal{D}$ *open* sets in the topological space. Sometimes, we call X a topological space if the topology follows from the context. A class $\mathcal{B}$ of open subsets of X is said to be a *basis* of X if every open subset of X is the union of some sets belonging to $\mathcal{B}$.

A class $\mathcal{B}_0$ of open subsets of X is said to be a *subbasis* if the class $\mathcal{B}$ composed of the empty set, the whole space X, and of the all intersections $B_1 \cap \cdots \cap B_n$, where $B_1, \ldots, B_n \in \mathcal{B}_0$, is a basis of X.

Observe that for every agent t, her knowledge base $\mathcal{E}_t$ may be treated as a subbasis of the topological space U, whereas the family $\mathcal{B}_t$, defined above may be conceived as a basis of the topological space U.

It is well known that

- Every empty set and the whole space are open.
- The intersection of two open sets is open.
- the union of arbitrary many open sets is an open set.

Let us suppose that Y is a subset of a topological space X. Consider the family of all open sets contained in Y. This family is non void, as the empty set and the union, denoted by $I(Y)$, of all elements of that family belong to it. It may be easily observed that $I(Y)$ is the greatest open set contained in Y. We call it the *interior* of the set Y.

The following theorem is well known:

Theorem 2.4 *For every class $\mathcal{B}_0$ of subsets of a set X there exists exactly one interior operation I in X such that $\mathcal{B}_0$ is a subbasis of the topological space X.*

Hence, by the above theorem a pair (X, I), may be also called a topological space, where I is an interior operation in X.

Now, we define the main notions of this paper.

Let $\mathcal{S}_t$ be a knowledge representation system of an agent t and let $\mathcal{E}_t$ be the knowledge base of t.

For every $X \subseteq U$ the set

$$I_t(X) = \bigcup_{[o]_t \subseteq X} [o]_t$$

is said to be the *partial knowledge* of the agent t about X determined

by $\mathcal{S}_t$.

Let us mention that the partial knowledge $I_t(X)$ is sometimes called positive knowledge of the agent t about X ([O], [R]) or the lower approximation of X,([P1]).

By the definition, the partial knowledge operator I_t is an interior operation and moreover, the pair (U, I_t) is a topological space and the family $\mathcal{E}_t$ is a subbasis of that space.

Thus we have:

Lemma 2.5 *For every agent t (or a group of agents t) and for all subsets X and Y of U the following holds:*

$$(1) \quad I_t(X \cap Y) = I_t(X) \cap I_t(Y).$$

$$(2) \quad I_t(X) \subseteq X.$$

$$(3) \quad I_t(X) \subseteq I_t I_t(X).$$

$$(4) \quad I_t(U) = U.$$

Put

$$C_t(X) = -I_t - (X) .$$

Thus, the operator C_t defined above is a closure operator in the topological space (U, I_t).

The next theorem follows from the definition of the partial knowledge operator I_t.

Theorem 2.6 *For every agent t and set $X \subseteq U$ the following holds:*

1. $I_t C_t(X) = C_t(X)$ *and* $C_t I_t(X) = I_t(X)$.

2. $I_t(X) = X$ *if and only if* $I_t(-X) = -X$.

We say that Y is *t-definable* if $I_t(Y) = Y$, that is, Y is t -definable if Y is an I_t-open set or, in other words, Y can be covered by elements of the knowledge base. Certainly, some sets can be covered by one knowledge base and not covered by another one. Thus, the set $I_t(Y)$ is the greatest subset of Y which is t-definable. Hence, the partial knowledge $I_t(Y)$ of t about Y is the collection of those objects which can be classified with full certainty by the agent t as elements of Y using attributes A_t.

The set $I_t(-Y)$ is the collection of those objects for which it can be determined without any ambiguity by the agent t, employing knowledge attributes A_t, that they belong to the complement of Y, that is, they do not belong to the set Y.

The set $I_t(-Y)$ is sometimes called negative knowledge ([O], [R]) about Y or the upper approximation of Y, ([P1]).

Recall every set may be defined as a collection of properties. Hence, every object $o \in I_t(Y)$ may be considered as a positive example of properties described by Y, that is, as a positive example of Y. Similarly, every object $o \in I_t(-Y)$ may be treated as a negative example of Y.

By the *knowledge of an agent t about Y*, determined by $\mathcal{S}_t$, we understand the set of objects which consists of all positive and negative examples of Y and denote it by $K_t(Y)$. Thus, for every $Y \subseteq U$

$$K_t(Y) = I_t(Y) \cup I_t(-Y) .$$

Let us notice that the set $K_t(Y)$ is an I_t-open set.

Example 4 Let t be singleton and let $\mathcal{S}_t$ be as in Example 3. Take as Y the set $\{o_1, o_2, o_4\}$. Then $I_t(Y) = \{o_1, o_4\}$, and $I_t(-Y) = \{o_5\}$. Hence, $K_t(Y) = \{o_1, o_4, o_5\}$.

3. PROPERTIES OF KNOWLEDGE OPERATORS I_t AND K_t

In this section we provide the basic properties of knowledge operators.

Let for every agent t (or group of agents t) a knowledge representation system $\mathcal{S}_t$ and knowledge base $\mathcal{E}_t$ be given. Recall that we have assumed that different agents perceive the same universe of discourse U.

Lemma 3.1 *Let t and s be different agents (or groups of agents). Let $\mathcal{S}_s$ and $\mathcal{S}_t$ be the corresponding knowledge representation systems and let X be a subset of the universe U. Then the following hold:*

1. $A_t \subseteq A_s$ *implies* $I_t(X) \subseteq I_s(X)$.
2. $\mathcal{E}_t \prec \mathcal{E}_s$ *if and only if* $I_s(X) \subseteq I_t(X)$.
3. $I_t(X) \cup I_s(X) \subseteq I_{t \cup s}(X)$.
4. $I_{t \circ s}(X) \subseteq I_t(X) \cap I_s(X)$.
5. $I_s I_t(X) \subseteq I_s(X)$ *and* $I_s I_t(X) \subseteq I_t(X)$.
6. $I_s I_t(X) \subseteq I_s(X) \cap I_t(X)$
7. $A_t \subseteq A_s$ *implies* $I_t I_s(X) = I_t(X)$.

We show now that the inclusions in 3–6 cannot be substituted by the

equalities. Indeed, take agents as in Example 3. Then for every $X \subseteq U$, $I_{s \cup w}(X) = X$. However, $I_s\{o_2, o_3, o_5\} = \{o_3\}$ and $I_w\{o_2, o_3, 0_5\} = \{o_5\}$. Hence, there is a set X such that the converse to (3) does not hold.

Notice, that for $X = \{0_2, 0_3\}$, $I_{tos}(X) = \emptyset$. However, $I_t\{o_2, o_3\} = \{o_2, o_3\}$ and $I_s\{o_2, o_3\} = \{o_3\}$. Thus, the converse to (4) is not true. Observe also that $I_t I_s\{o_2, o_3\} = \emptyset$, which shows that the converse to (5) and (6) is not valid.

It can be shown that if we impose the following conditions

$*_\cup$ $\quad \forall [o]_{t \cup s} \subseteq X \; (\exists [o_i]_t \subseteq X \; [o]_{t \cup s} \subseteq [o_i]_t \vee \exists [o_j]_s \subseteq X \; [o]_{t \cup s} \subseteq [o_j]_s)$.

$*_\circ$ $\quad \forall [o_i]_t \subseteq X \, \forall [o_j]_s \subseteq X \; [o_i]_t \cap [o_j]_s \neq \emptyset \, \exists [o]_{tos} \subseteq X \; [o_i]_t \cap [o_j]_s \subseteq [o]_{tos}$.

on a set of objects X then inclusions in (3), (4) and (5) can be replaced by equalities.

Immediately from Lemma 3.1 we obtain certain properties for knowledge operators. Namely,

Lemma 3.2 *For the same assumption as in Lemma 3.1 the following hold:*

1. $A_t \subseteq A_s$ *implies* $K_t(X) \subseteq K_s(X)$.
2. $\mathcal{E}_t \prec \mathcal{E}_s$ *if and only if* $K_s(X) \subseteq K_t(X)$.
3. $K_t(X) \cup K_s(X) \subseteq K_{t \cup s}(X)$.
4. $K_{tos} \subseteq K_t(X) \cap K_s(X)$.
5. *If* $\mathcal{E}_t \prec \mathcal{E}_s$ *then* $K_s K_t(X)$ *is an* I_t *open set.*
6. $K_t(X) = K_t(-X)$.
7. $I_t K_t(X) = K_t(X)$.

We say that the knowledge of t about X is *complete* if $K_t(X) = U$. Intuitively, if the knowledge of t about X is complete, then t with any doubt can decide for each object from the universe U whether it belongs to X or does not belong to X. Then the bordeline region is empty.

Lemma 3.3 *For every knowlegde operator* K_t *and set* X *the following hold:*

1. $K_t K_t(X) = U$.
2. $K_t \emptyset = U$.
3. $K_t U = U$.
4. $K_t I_t(X) = U$.

5. *If* $\mathcal{E}_t \prec \mathcal{E}_s$ *then* $K_tK_s(X) = U$.
6. $K_t(X) = U$ *if and only if* $I_t(X) = X$.

Notice that condition (5) says that knowledge of an agent t about knowledge of less efficient agent s is complete.

Observe that the knowledge operator is non-monotonic. Namely, the inclusion $X \subseteq Y$ does not need to imply that $K_t(X) \subseteq K_t(Y)$. Intuitively this fact is obvious as for every agent t the knowledge operator K_t is the union of two I_t open sets.

Finally, notice that certain paradoxes of epistemic logic are eliminated in our system, namely: *what the agent knows is true* and 'logical omniscience problem' which says that *an agent always knows all the consequences of her knowledge* ([Hi]).

Formally, we have:

Lemma 3.4 *For some knowledge operator* K_t *and some sets* X *and* Y *the following are not true:*

1. $K_t(X) \subseteq X$.
2. *If* $X \subseteq Y$ *and* $K_t(X) = U$ *then* $K_t(Y)$ *need not to be complete.*

We say that agents t and s reach consensus about X if $K_t(X) = K_s(X)$.

It is obvious that

- If X is simultaneously an I_t-open and an I_s-open set then agents t and s reach consensus about X.

4. t-LOGIC

In this section we are going to define the syntax and semantics of a logic for reasonig about knowledge of one agent or a group of agents. We call this logic t-logic.

We define first the formal language of t-logic.

Recall that for every agent t her knowledge operator K_t defined previously is expressed by a partial knowledge operator I_t interpreted as an interior operation. Thus, it is sufficient to describe the required language as a language containing a connective I_t, and then to define a connective K_t corresponding to the knowledge operator of t.

Let A_t be a set of attributes and $\{V_a\}_{a \in A_t}$ a family of finite sets. We associate with A_t and $\{V_a\}_{a \in A_t}$ a formal language $\mathcal{L}^t_{A_t,V_t}$ such that

all elements of A_t together with the set $V_t = \bigcup_{a \in A_t} V_a$ are treated as constants of $\mathcal{L}^t_{A_t,V_t}$, called *attribute constants* and *attribute-value constants*, respectively. By *atomic formulae* we mean pairs (a, v), where a is an attribute constant and v is an a-attribute value constant. The set of all formulae of $\mathcal{L}^t_{A_t,V_t}$ consists of all atomic formulae combined by means of binary propositional connectives $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, one unary propositional connective $-$ and one modal operator I_t. The modal operator I_t may be interpreted as an interior operation.

As axioms of t-logic we assume all axioms of classical logic enriched by axioms for the unary operator I_t and some specific axioms which express characteristic properties of knowledge representation systems.

We will use the following abbreviations: for any formula ϕ

$$\phi \wedge -\phi =_{df} \bot \text{ and } \phi \vee -\phi =_{df} \top.$$

Axioms for I_t are as follows: for any formulae ϕ and ψ

(i_1) $\quad I_t\phi \rightarrow \phi.$

(i_2) $\quad I_t\phi \rightarrow I_tI_t\phi.$

(i_3) $\quad I_t(\phi \vee \psi) \leftrightarrow (I_t\phi \vee I_t\psi).$

(i_4) $\quad I_t\top.$

The specific axioms of t-logic are as follows:

1. $(a, v) \wedge (a, u) \leftrightarrow \bot$ for any $a \in A_t$, $v, u \in V_a$ and $v \neq u$.
2. $\bigvee_{v \in V_a}(a, v) \leftrightarrow \top$, for every $a \in A_t$,
3. $-(a, v) \leftrightarrow \bigvee\{(a, u) : u \in V_a, u \neq v\}$.

The motivation of the axioms above is the following: Specific axioms should be characteristic for our notion of the knowledge representation system.

Observe that the first axiom follows from the assumption that each object can have exactly one value for each attribute.

Axiom (2) follows from the assumption that each object in $\mathcal{S}_t$ has a value with respect to every attribute. Hence, the description of objects is complete up to a given set of attributes. In other words, for every $a \in A_t$ and every object x the entry in the row x and the column a (in $\mathcal{S}_t$ viewed as a table) is nonempty.

The third axiom allows us to figure out negation in such a way that instead of saying that an object does not possess a given property we can say that it has one of the remaining properties. For example, instead

of saying that something is not blue we can say it is either red or green or yellow, etc.

We say that a formula ϕ is *derivable in t-logic from a set of formulae* F, denoted by $F \vdash \phi$, provided it can be concluded from F by means of the axioms, *modus ponens* and the following rule:

$$\frac{\phi}{I_t\phi}\,.$$

If ϕ is derivable from the empty set, then we write $\vdash \phi$ and say ϕ *is derivable*. Clearly, all classical tautologies are derivable. Also, $\vdash \top$ and $\nvdash \bot$.

In the standard way we may prove the Deduction Theorem:

Theorem 4.1 $F \vdash \phi \rightarrow \psi$ *if and only if* $F \cup \{\phi\} \vdash \psi$.

It is also not difficult to prove

Lemma 4.2 *The set of all specific axioms is dependent. Namely, for every different* $i, j = 1, 2, 3,\ \{i, j\} \vdash k,\ k = 1, 2, 3.$

Intuitively speaking, formulae of t-logic are meant as a description of objects of the universe. Clearly, some objects may have the same description. Thus formulae may describe subsets of objects obeying properties expressed by these formulae. For instance, a natural interpretation of an atomic formula (a, v) is the set of all objects having value v for the attribute a. Hence, a natural interpretation of a formula of the form $I_t(a, v)$ is the set of all objects which an agent t recognizes as objects having the property v for the attribute a.

Now, we are going to give semantics for the language of t-logic in the style of Tarski.

Let $\mathcal{L}^t_{A_t,V_t}$ be the language described above determined by given sets A_t and $V_t = \bigcup_{a\in A_t} V_a$. From now on, we will assume that every knowledge representation system under consideration is of the form $\mathcal{S}_t = (U, A_t)$ and that for every $a \in A_t$, the domain of a is the set V_a. Recall that $\mathcal{E}_t = \{[x_1]_{A_t}, \ldots, [x_l]_{A_t}\}$ is a partition of the universe U given by $ind(A_t)$.

Let ϕ be a formula from $\mathcal{L}^t_{A_t,V_t}$. We say that an object x *satisfies a formula* ϕ in $\mathcal{S}_t$, denoted by $x \models_{\mathcal{S}_t} \phi$ or in short $x \models \phi$ if and only if the following conditions are satisfied:

1. $x \models (a, v)$ if and only if $a(x) = v$,

2. $x \models -\phi$ if and only if $x \not\models \phi$,

3. $\models \phi \vee \psi$ if and only if $x \models \phi$ or $x \models \psi$,

4. $x \models \phi \wedge \psi$ if and only if $x \models \phi$ and $x \models \psi$,

5. $x \models \phi \rightarrow \psi$ if and only if $x \models -\phi \vee \psi$,

6. $x \models \phi \leftrightarrow \psi$ if and only if $x \models \phi \rightarrow \psi$ and $x \models \psi \rightarrow \phi$,

7. $x \models I_t\phi$ if and only if for all $x_i \in U$ if $x_i \in [x]_{A_t}$ then $x_i \models \phi$.

For any formula ϕ the set $|\phi|_{\mathcal{S}_t}$, defined by

$$|\phi|_{\mathcal{S}_t} = \{x \in U : x \models \phi\},$$

will be called the *meaning of the formula* ϕ in $\mathcal{S}_t$. We will omit the subscript $\mathcal{S}_t$, if the knowledge representation system $\mathcal{S}_t$ follows from the context.

In the next lemma we list all basic properties of the meaning of formulae.

Lemma 4.3 *For every atomic formula* (a, v) *and any formulae* ϕ *and* ψ *the following hold:*

1. $|(a, v)| = \{x \in U : a(x) = v\}$.

2. $|-\phi| = -|\phi|$.

3. $|\phi \vee \psi| = |\phi| \cup |\psi|$.

4. $|\phi \wedge \psi| = |\phi| \cap |\psi|$.

5. $|\phi \rightarrow \psi| = -|\phi| \cup |\psi|$.

6. $|\phi \leftrightarrow \psi| = |\phi \rightarrow \psi| \cap |\psi \rightarrow \phi|$.

7. $|I_t\phi| = \bigcup_{[x_i]_{A_t} \subseteq |\phi|} [x_i]_{A_t}$.

Clearly, by the definition of the partial knowledge operator I_t of t (Section 2.2) condition 7 may be rewritten in the form:

$$|I_t\phi| = I_t|\phi|,$$

where I_t on the left hand side of the above equality is the logical connective and I_t on the right hand side of the equality is the partial knowledge operator of an agent t determined by $\mathcal{S}_t$.

Finally, we say that a formula ϕ not proceeded by I_t is *true in* $\mathcal{S}_t$, denoted by $\models_{\mathcal{S}_t} \phi$, if $|\phi|_{\mathcal{S}_t} = U$. A formula of the form $I_t\phi$ is *true in* $\mathcal{S}_t$ if

$$|I_t\phi \vee I_t - \phi|_{\mathcal{S}_t} = U.$$

If a formula ϕ is true in $\mathcal{S}_t$, then we call $\mathcal{S}_t$ a model for ϕ.

We say that F *implies* ϕ, denoted by $F \models \phi$, if every model of F is a model of ϕ.

The next two theorems show that our axiomatization has been adequatly choosen and that it is complete. Namely,

Theorem 4.4 *(soundness) Let F be a set of formulae. If $F \vdash \phi$ then $F \models \phi$.*

Theorem 4.5 *(completeness) Let F be a set of formulae and let ϕ be a formula of $\mathcal{L}^t_{A_t,V_t}$. If $F \models \phi$ then $F \vdash \phi$.*

Define now a new modal connective K_t as follows: for any formula ϕ put

$$K_t\phi \equiv I_t\phi \vee I_t - \phi\,.$$

By 4.3 we can show that

$$|K_t\phi| = K_t|\phi|\,,$$

where K_t on the left hand side is the unary connective defined above, and K_t on the right hand side of the equality is the knowledge operator determined by the knowledge representation system $\mathcal{S}_t$.

Theorem 4.6 *Knowledge of an agent t about set of objects X is complete, that is $K_t(X) = U$ if and only if a formula of the form $I_t\phi$ is true in $\mathcal{S}_t$, where the meaning of ϕ is X, that is, $|\phi|_{\mathcal{S}_t} = X$, which is equivalent to $|I_t\phi|_{\mathcal{S}_t} = X$.*

5. T-LOGIC

Let T be a family of agents and let for every $t \in T$, $\mathcal{L}^t_{A_t,V_t}$ be a formal language defined in the previous section.

Put

$$A = \bigcup_{t\in T} A_t \text{ and } V = \bigcup_{t\in T} V_t\,.$$

We associate with T, A and V a formal language $\mathcal{L}^T_{A,V}$ which will enable us to express facts about the knowledge of sets of agents.

The language $\mathcal{L}^T_{A,V}$ consists of two levels. The expressions of these levels are called *terms* and *formulae*, respectively. Intuitively, terms describe relationships between agents, whereas formulae express certain facts concerning sets of objects and knowledge of $t \in T$ about these sets.

Terms are built up from elementary parts, called *variables*, two constants **0** and **1** and operations: $\cup, \circ, \Rightarrow$. More precisely, the *set of all terms* is defined to be the least set T with the following three properties:

- **0** and **1** are in T.
- all variables are in T,
- $t \cup s$, $t \circ s$ and $t \Rightarrow s$ are in T whenever t, s are in T.

We sometime write $t \cup s$ as ts if no confusion is created.

The *set of all formulae* F is the smallest set containing all atomic formulae which are of the form (a, v), where $a \in A$ and $v \in V$ and it is closed with respect to propositional connectives $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $-$ and a family $\{I_t\}_{t \subseteq T}$ of modal connectives, that is, the following conditions are satisfied for F:

- $(a, v) \in F$ for every $a \in A$ and $v \in V$.
- $\phi \vee \psi$, $\phi \wedge \psi$, $\phi \rightarrow \psi$ $\phi \leftrightarrow \psi$, and $-\phi$ are in F, whenever ϕ and ψ are in F.
- For every $t \in T$ $I_t, \phi \in F$ whenever ϕ is in F.

Axioms for T-logic consist of two groups: one for terms and one for formulae.

Axioms for terms are as follows:

(t_1) $(t \cup s) \Rightarrow (s \cup t)$ and $t \circ s \Rightarrow (s \circ t)$.

(t_2) $((t \cup s) \cup u) \Rightarrow (t \cup (s \cup u))$ and $((t \circ s) \circ u) \Rightarrow (t \circ (s \circ u))$.

(t_3) $(t \cup (t \circ s)) \Rightarrow t$ and $s \Rightarrow (s \circ (s \cup t))$.

Axioms for formulae are as follows:

(f_1) all axioms for classical logic,

(f_2) axioms 1–3 for t-logic where instead of A_t take A,

(f_3) for every term t, all axioms i_1–i_4.

We say that a formula ϕ is *derivable* in T-logic from a set of formulae F, denoted by $F \vdash_T \phi$ provided it can be concluded from F by means of the axioms and the following rules: modus ponens, and

$$\frac{\phi}{I_t\phi}$$

$$\frac{t \Rightarrow s}{I_t\phi \to I_s\phi},$$

where I_t is any modal connective, ϕ any formula and t and s any terms.

Lemma 5.1 *The following are examples of derivable formulas in T-logic from empty set of formulae:*

1. $(I_t\phi \vee I_s\phi) \to I_{t\cup s}\phi$,

2. $I_{t\circ s}\phi \to (I_t\phi \wedge I_s\phi)$,

3. $I_tI_s\phi \to I_t\phi$,

4. $I_tI_s\phi \to I_s\phi$.

Let for every I_t and formula ϕ

$$K_t\phi \equiv I_t\phi \vee I_t - \phi\,.$$

Thus we have defined a new modal connective in the language $\mathcal{L}^T_{A,V}$ which may be interpreted as the knowledge operator of an agent t.

Lemma 5.2 *The following formulas are derivable in T-logic: for every knowledge operator K_t and formula ϕ*

1. $(K_t\phi \vee K_s\phi) \to K_{t\cup s}\phi$,

2. $K_{t\circ s}\phi \to (K_t\phi \wedge K_s\phi)$,

3. $K_tK_t\phi$,

4. $K_t\phi \to K_t - \phi$,

Lemma 5.3 *The following are not provable in T-logic:*

1. $K_t\phi \rightarrow \phi$.

2. $(\phi \rightarrow \psi) \wedge K_t\phi \rightarrow K_t\psi$.

We are going now to describe semantics of T-logic.

Let $\mathcal{L}_{A,T}^{T}$ be the language describe above. Let $\mathcal{S}_T = (U, A)$ be a knowledge representation system of a group of agents T. Let for every $t \in T$, $\mathcal{E}_t$ be a partition of U determined by a subset A_t of A. Recall that a T-algebra is an algebra of the form $(T, \cup, \circ, \Rightarrow, \mathbf{0}, \mathbf{1})$.

We will interpret now all terms as elements of a T-algebra, that is, the identity mapping id is an isomorphism of the algebra of terms onto T-algebra. We say that *a term t is true in T-algebra* if $id(t) = \mathbf{1}$.

Immediately from the above definition we obtain:

Theorem 5.4 *For every term t of T-logic, t is derivable if and only if t is true in T-algebra.*

We say that an object x *satisfies a formula* ϕ in $(\mathcal{S}_T, \{\mathcal{E}_t\}_{t\in T})$ if conditions 1–6 from Section 4 are satisfied and condition 7 holds for every connective I_t. A formula ϕ not proceeded by any I_t is *true in* $(\mathcal{S}_T, \{\mathcal{E}_t\}_{t\in T})$ if $|\phi|_{\mathcal{S}_T} = U$. We say that a formula of the form $I_t\phi$ is *true in* $(\mathcal{S}_T, \{\mathcal{E}_t\}_{t\in T})$ if

$$|I_t\phi \vee I_t - \phi|_{\mathcal{S}_T} = U \; .$$

If a formula is true in $(\mathcal{S}_T, \{\mathcal{E}_t\}_{t\in T})$ then we say that $\mathcal{S}_T$ is a *model* for that formula.

We say that a set of formulae F *implies a formula* ϕ, denoted by $F \models_T \phi$, if every model for F is a model for ϕ.

It can be shown:

Theorem 5.5 (**soundness**) *Let F be a set of formulae. If $F \vdash_T \phi$ then $F \models_T \phi$.*

Theorem 5.6 (**completeness**) *Let F be a set of formulae and let ϕ be a formula of $\mathcal{L}_{A,V}^{T}$. If $F \models_T \phi$ then $F \vdash_T \phi$.*

Finally we have

Theorem 5.7 *$S_T = (U, A)$ is a model for a formula ϕ if and only if there is a subset A_t of A such that $S_t = (U, A_t)$ is a model for ϕ.*

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TOMASZ SKURA

CHARACTERIZING PROPOSITIONAL LOGICS BY FORMULAS

INTRODUCTION

I say that a set S of formulas characterizes a logic L iff for every formula F we have

> F is not a theorem of L iff for some $G \in S$, G is derivable in L from F.

Yes. This is another kind of completeness theorem. The reason I consider this hackneyed problem here is that this syntactic method, apart from being a very nice way of describing a logic, is in a sense more general than the semantic one. Let us take the classical propositional logic as an example. It is easy to see that the constant $\perp$ (falsity) is characteristic for this logic. Thus $\perp$ is a syntactic counterpart of the two-element Boolean algebra. The aim of this paper is to find such characterizations for some of the well-known propositional logics. In fact the formulas I give are syntactic counterparts of the finite models characterizing these logics. However let me point out that there are logics that cannot be characterized by finite models but do have recursive syntactic descriptions of this kind (cf. [2], [3]).

In what follows, simple proofs will be omitted. (Some of those proofs can be found in [4]).

1. JAŚKOWSKI FORMULAS

The symbol FOR will denote the set of all formulas generated from the set VAR of propositional variables by the connectives $\wedge$, $\vee$, $\rightarrow$, and the constant $\perp$. As usual $\top = \perp \rightarrow \perp$ and $\neg F = F \rightarrow \perp$ for any $F \in$ FOR.

We define the sequence of Jaśkowski formulas $J_1, J_2, \ldots$

Jan Woleński (ed.), Philosophical Logic in Poland, 239–245.

DEFINITION 1.1.

1. $G_1^i = p_i \qquad (i \geq 1)$

2. For $n > 1, i \geq 1$,
$$G_n^i = \bigvee \{\neg\neg G_{n-1}^s : ni - n + 1 \leq s \leq ni\}.$$

Moreover,

1. $D_1^i = \top \qquad (i \geq 1)$

2. For $n > 1, i \geq 1$,
$$D_n^i = \bigwedge \{D_{n-1}^r : ni - n + 1 \leq r \leq ni\} \wedge$$
$$\bigwedge \{\neg (G_{n-1}^s \wedge G_{n-1}^r) : ni - n + 1 \leq s \neq r \leq ni\} \wedge$$
$$\bigwedge \{(\neg\neg G_{n-1}^s \to G_{n-1}^s) \to (\neg\neg G_n^i \to G_n^i) : ni - n + 1 \leq$$
$$\leq s \leq ni\}.$$

Finally
$$J_n = D_n^1 \to \left(\neg\neg G_n^1 \to G_n^1\right).$$

In this section the characters $\mathbf{A}$, $\mathbf{B}$ will denote Heyting algebras. The symbol $\mathbf{A} + \mathbf{B}$ denotes the result of putting $\mathbf{B}$ on top of $\mathbf{A}$ and identifying 0_B with 1_A. The sequence of Jaśkowski algebras $\mathbf{A}_0$, $\mathbf{A}_1$, ... is defined as follows: $\mathbf{A}_0$ is the two-element Boolean algebra, and $\mathbf{A}_n = \mathbf{A}_{n-1}^n + \mathbf{A}_0$ for $n \geq 1$.

We also write $\mathbf{A}+$ instead of $\mathbf{A}+\mathbf{A}_0$. Every Jaśkowski algebra $\mathbf{A}_n$ is strongly compact, i.e. there is a greatest element in $A_n - \{1_n\}$ (denoted by $*_n$). As usual π_j is the projection map on the j-th coordinate of $\prod_{i \in I} B_i$. The symbol INT will denote the set of all theorems of the intuitionistic propositional logic.

Moreover, for any $n, i \geq 1$ we define the function $f_n^i : A_n \to \mathrm{FOR}$ as follows:

1. $$f_1^i(x) = \begin{cases} \neg\neg p_i & \text{if } x = 1_1 \\ p_i & \text{if } x = *_1 \\ \bot & \text{if } x = 0_1 \end{cases} \qquad (x \in A_1)$$

2. For $n > 1$,
$$f_n^i(x) =$$
$$\begin{cases} \neg\neg G_n^i & \text{if } x = 1_n \\ f_{n-1}^{ni-n+1}(x_1) \vee \ldots \vee f_{n-1}^{ni}(x_n) & \text{if } x = (x_1, \ldots, x_n) \end{cases} \quad (x \in A_n)$$

LEMMA 1.2. Let v be a valuation in $\mathbf{A}_m$, $m \geq 1, n > 1$.

If $v\left(D_n^i\right) = 1_m$ and $x, y \in A_n - \{1_n\}$ then $v\left(f_n^i(x_1) \rightarrow f_n^i(y)\right) =$

$$v\left(\left(f_{n-1}^{ni-n+1}(x_1) \rightarrow f_{n-1}^{ni-n+1}(y_1)\right) \wedge \ldots \wedge \left(f_{n-1}^{ni}(x_n) \rightarrow f_{n-1}^{ni}(y_n)\right)\right).$$

Proof. By the following facts:

(1) $f_n^i(x) \rightarrow \neg\neg G_n^i \in \mathrm{INT} \qquad (x \in A_n, n \geq 1)$

(2) $\neg(F \wedge G) \rightarrow \neg(\neg\neg F \wedge \neg\neg G) \in \mathrm{INT}$

(3) $(E \rightarrow F) \wedge \neg(F \wedge G) \rightarrow \neg(E \wedge G) \in \mathrm{INT}$

(4) $\bigwedge\{\neg(E_i \wedge F_j) : 1 \leq i \neq j \leq k\} \rightarrow$
$((E_1 \vee \ldots \vee E_k \rightarrow F_1 \vee \ldots \vee F_k) \rightarrow$
$\bigwedge\{E_i \rightarrow F_i : 1 \leq i \leq k\}) \in \mathrm{INT}$

LEMMA 1.3. Let v be a valuation in $\mathbf{A}_m$, $m, n \geq 1$.

If $v\left(D_n^i\right) = 1_m, v\left(\neg\neg G_n^i \rightarrow G_n^i\right) \neq 1_m$ then for all $x, y \in A_n$,

$v \circ f_n^i(x) \leq v \circ f_n^i(y)$ implies $x \leq y$.

Proof. Simple.

For any Heyting algebra $\mathbf{A}$ we define the function $\pi_* : A+ \rightarrow A$ as follows:

$$\pi_*(a) = \begin{cases} a & \text{if } a \in A \\ 1_A & \text{otherwise} \end{cases} \qquad (a \in A+)$$

LEMMA 1.4. Let h be a function from $A+$ into $B+$ such that

$h(x) \leq h(y)$ implies $x \leq y$ for all $x, y \in A+$,

$h(1_{A+}) = 1_{B+}$,

and $\pi_* \circ h$ is a homomorphism of $\mathbf{A}+$ into $\mathbf{B}$. Then h is a homomorphism.

Proof. Routine.

LEMMA 1.5. Let v be a valuation in $\mathbf{A}_m$, $m, n \geq 1$.

If $v\left(D_n^i \wedge \neg\neg G_n^i\right) = 1_m$ then $v \circ f_n^i$ is a homomorphism of $\mathbf{A}_n$ into $\mathbf{A}_m$.

Proof. By induction on (n, m).

1. $n = m = 1$. Obvious.

2. $n > 1$ or $m > 1$. Assume that
$v\left(D_n^i \wedge \neg\neg G_n^i\right) = 1_m$.

Case 1. $v(G_n^i) = 1_m$.
We may also assume that $n > 1$.

Then $v\left(\neg\neg G_{n-1}^s\right) = 1_m$ for some $ni - n + 1 \leq s \leq ni$.
Also $v(D_{n-1}) = 1_m$.

Hence by the inductive hypothesis $v \circ f_{n-1}^s$ is a homomorphism of $\mathbf{A}_{n-1}$ into $\mathbf{A}_m$. Thus in view of the fact that $v\left(G_{n-1}^r\right) = 0_m$ for $ni - n + 1 \leq s \neq r \leq ni$ it is easy to check that $v \circ f_n^i$ is indeed a homomorphism.
Case 2. $v(G_n^i) \neq 1_m$.

If $m = 1$ then $n > 1$ and $v(G_n^i) = *_1$, whence $v\left(\neg\neg G_{n-1}^s\right) = *_1$, for some s, which is impossible, so we assume that $m > 1$.

Since $v\left(\neg\neg G_n^i \rightarrow G_n^i\right) \neq 1_m$, by Lemma 1.3 we have that $v \circ f_n^i(x) \leq v \circ f_n^i(y)$ implies $x \leq y$. Thus in view of Lemma 1.4 it suffices to show that $v_* \circ f_n^i$, where $v_* = \pi_* \circ v$, is a homomorphism of $\mathbf{A}_n$ into $\mathbf{A}_{m-1}^m$.

Now $v_t\left(D_n^i \wedge \neg\neg G_n^i\right) = 1_{m-1}$ for every $1 \leq t \leq m$, where $v_t = \pi_t \circ v_*$. Hence by the inductive hypothesis for every $1 \leq t \leq m$, $v_t \circ f_n^i$ is a homomorphism of $\mathbf{A}_n$ into $\mathbf{A}_{m-1}$. Therefore $v_* \circ f_n^i$ is a homomorphism. QED.

LEMMA 1.6. Let v be a valuation in $\mathbf{A}_m$, $m, n \geq 1$.

If $v\left(D_n^i \wedge \neg\neg G_n^i\right) = 1_m, v(G_n^i) \neq 1_m$ then $v \circ f_n^i$ is an embedding of $\mathbf{A}_n$ into $\mathbf{A}_m$.
Proof. By Lemmas 1.5 and 1.3.

THEOREM 1.7. Let $F \in$ FOR. Then $F \notin$ INT iff there is $n \geq 1$ such that $e(F) \rightarrow J_n \in$ INT for some substitution e.
Proof. ($\leftarrow$) It is easily seen that each J_n is not a tautology of the algebra $\mathbf{A}_n$.
($\rightarrow$) Assume that $F \notin$ INT. Then by Jaśkowski's Completeness Theorem

there exists $n \geq 1$ such that F is not a tautology of the algebra $\mathbf{A}_n$, i.e. $w(F) \neq 1_n$ for some valuation w in $\mathbf{A}_n$.

Now let e be a substitution such that $e(p) = f_n^1\left(w(p)\right)$ for any $p \in$ VAR. Using Lemma 1.6 it is not difficult to show that $e(F) \to J_n \in$ INT (cf. [3]). QED.

2. HEYTING SETS OF FORMULAS

In this section we define sets of formulas that correspond to countable Heyting algebras.

We write "$S \subseteq_f T$" instead of "S is a finite subset of T".

DEFINITION 2.1. Let $X = Y \cup \{\bot, \top\}$, where $Y \subseteq$ VAR, and let g be a function from $\{a\#b : a, b \in X, \# \in \{\wedge, \vee, \to\}\}$ into X. Then we define the following sets of formulas:

$$\begin{aligned} W(X,g) &= \{a\#b \equiv g(a\#b) : a, b \in X, \# \in \{\wedge, \vee, \to\}\}, \\ Z(x,g) &= \left\{\bigwedge S \to p : S \subseteq_f W(X,g), p \in X - \{\top\}\right\}, \end{aligned}$$

and we say that $Z(X,g)$ is a Heyting set iff for all $a, b \in X$ the following conditions are satisfied:

$$g(a\#b) = g(b\#a) \qquad g\left(a\#g(b\#c)\right) = g\left(g(a\#b)\#c\right)$$

$$g(a\#a) = a \qquad \text{where } \# \in \{\wedge, \vee\}$$

$$a = g\left(a \vee g(a \wedge b)\right) \qquad a = g\left(a \wedge g(a \vee b)\right)$$

$$g\left(a \wedge g(b \vee c)\right) = g\left(g(a \wedge b) \vee g(a \wedge c)\right)$$

$$g(a \wedge \bot) = \bot \qquad g(a \vee \top) = \top \qquad g(a \to a) = \top$$

$$g\left(g(a \to b) \wedge b\right) = b \qquad g\left(a \wedge g(a \to b)\right) = g(a \wedge b)$$

$$g\left(a \to g(b \wedge c)\right) = g\left(g(a \to b) \wedge g(a \to c)\right)$$

$$g\left(g(a \vee b) \to c\right) = g\left(g(a \to c) \wedge g(b \to c)\right).$$

THEOREM 2.2. Let $F \in$ FOR. Then $F \notin$ INT iff there is a finite Heyting set Z such that $e(F) \to G \in$ INT for some $G \in Z$ and for some substitution e.

Proof. Simple.

3. MODAL FORMULAS

Let FOR be the set of all formulas generated from the set VAR by the connectives $\neg, \wedge, \vee, \rightarrow, \Box$.

S5 FORMULAS

Following Segerberg [1] we define the following sequence of formulas:

$$K_n = \Box p_1 \vee \Box(p_1 \rightarrow p_2) \vee \ldots \vee \Box(p_1 \wedge \ldots \wedge p_{n-1} \rightarrow p_n)$$
$$(n \geq 2).$$

THEOREM 3.1. Let $F \in$ FOR. Then $F \notin$ S5 iff there is $n \geq 2$ such that $\Box e(F) \rightarrow K_n \in$ S5 for some substitution e.
Proof. Simple.

S4 FORMULAS

Let $S_n = \{E_1, \ldots, E_n\}$, where $E_1 = p_1, E_2 = p_1 \rightarrow p_2, \ldots,$

$E_{n-1} = p_1 \wedge \ldots \wedge p_{n-2} \rightarrow p_{n-1},$

$E_n = \neg(p_1 \wedge \ldots \wedge p_{n-1}) \qquad (n \geq 2).$

DEFINITION 3.2. Let g be a function from S_n into 2^{S_n}, $n \geq 2$. Then $L_n(g) := \Box \bigwedge \{\Box E_i \equiv \bigwedge g(E_i) : 1 \leq i \leq n\} \rightarrow K_n$, and we say that $L_n(g)$ is an S4 formula iff for every $1 \leq i \leq n$ the following conditions are satisfied:

$$E_i \in g(E_i),$$

$$\bigcup \{g(H) : H \in g(E_i)\} \subseteq g(E_i).$$

THEOREM 3.3. Let $F \in$ FOR. Then $F \notin$ S4 iff there is an S4 formula $L_n(g)$ such that $\Box e(F) \rightarrow L_n(g) \in$ S4 for some substitution e.
Proof. Simple.

4. ŁUKASIEWICZ SEQUENTS

In order the characterize a consequence relation we can use sequents rather than formulas. This approach has here the additional advantage of avoiding certain uncomfortable syntactic properties of Łukasiewicz logics.

Let FOR be the set of all formulas generated from VAR by the connectives $\neg, \rightarrow$. Also let $F \vee G = (F \rightarrow G) \rightarrow G$, $F \wedge G = \neg(\neg F \vee \neg F)$, and $F \equiv G = (F \rightarrow G) \wedge (G \rightarrow F)$. A sequent is a pair

(X, F), where $X \cup \{F\} \subseteq_f$ FOR.

Let $\vdash$ be the consequence relation induced by the axioms of the infinite-valued Łukasiewicz logic and the modus ponens rule. For any $X \cup Y \cup \{F, G\} \subseteq_f$ FOR we say

$$(X, F) \mid\!\vdash (Y, G)$$

(read "(Y, G) is $\vdash$-derivable from (X, F)") iff there is a finite sequence $F_1, \ldots, F_k$ of formulas such that $F_k = G$ and for each $1 \leq i \leq k$, either $F_i \in Y$ or F_i is obtained from some preceding formulas by one of the following rules:

$$\{(S, E) \text{ is a sequent: } S \vdash E\},$$

$$\{(eX, eF) : e \text{ is a substitution}\}$$

As usual $F \rightarrow_0 G = G$ and $F \rightarrow_{n+1} G = F \rightarrow (F \rightarrow_n G)$.

THEOREM 4.1. Let $X \cup \{F\} \subseteq_f$ FOR. Then $X \nvdash F$ iff for some $n \geq 0$ we have $(X, F) \mid\!\vdash (\{(p \rightarrow_n \neg p) \equiv p\}, p)$.

Proof. Simple.

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ROMAN SUSZKO

THE REIFICATION OF SITUATIONS*

At the conference of the Logic Section of IFiS PAN in October 1969, Professor J. Słupecki introduced a certain formal system W, which was published in the article 'A Generalization of Modal Logic'.[1] This paper was the point of departure for the following considerations.

Słupecki's article contains a certain idea with which the author hardly sympathizes, as can be supposed from some of his formulas. He proposes, namely, to read the expressions $p * x$ as follows:

(1) saying that p, we state (the event) x;
(2) sentence p states the event x.

On the other hand, Słupecki's work succeeds in my opinion in the idea of formalization of what is called the reification of situations. Such an idea suggests itself in the reading of this work, as we treat sentential variables p, q, r ... and name variables x, y, z, ... analogously as running over a certain universe (the universe of situations) as well as the universe of objects), and we use the star in expressions of the form $p * x$ as a symbol of a certain primitive undefined relation between the situation (that) p and object x or – in other words – between that, which the sentence on the left-hand side of the star describes, and that which the name on the right-hand side designates. Słupecki assumes that exactly one object x corresponds (in the sense of $*$) to each situation p. The star is therefore a symbol of a certain function of (abstract) objects, known here as 'events', to any given situation. This function is, according to definition D1 and axiom A, one-to-one homomorphism of the structure of the universe of situations determined by sentential connectives on the structure of the body of experiences, which in Słupecki's work constitutes a Boolean algebra. The function $*$ is, therefore, an isomorphism between the universe of situations and the algebra of events.

An abstract algebra of events is for many a very intuitive creation and certainly more natural than the universe of situations, which seems to be a suspicious hypostasis. Such an opinion can be understood from

Jan Woleński (ed.), Philosophical Logic in Poland, 247–270.

Słupecki's suggestion as to how to read the formula $p * x$.

I do not think that this point of view is sound. I am rather inclined to treat situations as primary creations, and to think that abstract objects like events are the product of a certain abstract process, whose formal form is the function $*$ introduced by Słupecki. I call this precisely 'the reification of situations'. Below I formulate the basic formal characteristics of the reification of situations and treat certain issues arising therein.

My considerations are not based directly on the system W of Słupecki. The foundation of my construction is non-Fregean logic with an identity connective and an identity predicate. I do not avail myself of Słupecki's special assumptions, such as:

(a) events constitute a Boolean algebra
(b) situations (that) $p * x$ are reified (via $*$) into the unity of zero of a Boolean algebra of events.

Instead of Słupecki's star, which is formator of the category: sentence/sentence, name, I use the symbol of reification the formator $\mathbf{T}$ of the category: name/sentence, which is read as follows $\mathbf{T}(p)$ = the reification of the situation (that) p = an event (abstract object) corresponding to the situation (that) p.

In order not to complicate the work too much, I do not use quantifiers. The formalized language containing the theory of the reification of situations is therefore open (that is to say, without bound variables). If one omits precisely this circumstance, my construction seems more general than the Słupecki system. That system can be reconstructed by certain minimal modifications of the foundation of the theory introduced below.

The great task of the theory of reification is to show in what way the so-called ontological assumptions of the structure of the universe of situations are transferred to eventas by reifications, and to impose an algebraic structure on them. Such an approach to the theory of reification flows from the earlier expressed opinion that situations are primary and events are derived. One should not confuse this point of view with the false opinion, I believe, that situations are primary in relation to all objects. It is an altogether different and difficult matter, and in this case a certain consultation of Wittgenstein would be very useful. But the fact that situations are primary in relation to their reification is rather natural. The abstract process, of which the formal expression is the reification of a situations, finds – I think – its fragmentary expression in ordinary

language; I write 'I think' since I enter into the competence of linguists. These examples given by Słupecki are an illustration. Thus, forest fire = reification of the situation that the forest is burning, and Matt's death = reification of the situation that Matt died. These examples do not give evidence that an explicit symbol of the reification of situations, corresponding to the star of Słupecki of our $\mathbf{T}$, exists in natural language. They are examples giving evidence that the grammatical apparatus of a natural language can often, though certainly not always, transform sentences p (describing situations) into names x (designating particular events) such that $x = \mathbf{T}(p)$, and sentences containing those names. The opposite transformation is something unnatural, and is hardly taken into consideration by grammarians.

This observation obviously does not remove the most serious difficulties which appear in connection with situations. The principal difficulty appears at the moment of incorporation of non-trivial theories written in natural language with help of (bound) sentence variables. Reading formulas appearing in this theory in natural language immediately raises serious doubts for many logicians with regard to sense or correctness. There are no such difficulties, or they are considerably less, in the reading of formulas with name variables (not sentential. It is probably the symptom of some deep, historical attribute of our thought and natural language, whose examination and explication will certainly be prolonged and arduous.

From the rather narrow point of view of formal logic the following considerations are suggested. The difference between a sentence and a name is not exhausted in their syntactical properties. A certain syntactic analogy even exists between the category of sentences and the category of names, which can stretch very far (for example the rules of operations for quantifiers are formally similar in the case of sentential and name variables). The difference between sentences and names appears first of all in that sentences, and not names, are subject to assertion, as well as that sentences are premises and conclusions in reasoning. These distinctions on the language level are transferred in some manner to that to which the sentences and names semantically refer. Semantical relation (reference), however, of sentences and names are also – formally – to a certain degree analogical.

Names designate and sentences describe. The difference in terminology (designate, describe) is not essential. The essential point is that we attribute reference to something both to names and to sentences, and

that this, in the case of a given name and a given sentences, is exactly one; with the assumption, obviously, of a univocal sense of expression and with exclusion of mythological terms.

This analogy, however, is not complete, just like the analogy between sentences and names, with result that a categorial gap exists between that which a sentence describes (a situation) and that which a name designates (an object). The fact that the expressions $p = x$ and $p \neq x$, where on the left we have a sentence and on the right a name, are not well formed formulas, shows this profound gap.

The analogies between situation and objects as well as that between sentences and names are not complete. But it does not stop at the level of the formation of sentences and names, not at the level of the formal operation on them in accordance with logic. What, therefore, is the cause that our thought and natural language discriminate sentence variables to a certain degree, and particularly, general and existential sentences about situations?

The above considerations about the reification of situations show that the theory of situations and the theory of events are, in certain manner, equivalent. Why, therefore, prefer the theory of events to the theory of situations?

1.

Reification is not a translation from one language to a second. The theory of reification is also not formulated in the metalanguage. It is theory in which one speaks about situations and objects. Reification is a one-to-one mapping of the universe of situations into an algebra of certain objects, called events. The task of the theory of reification is the introduction of operations of transfer (via reification) of the properties of universe of situations to events. The results of the theory of reification are straightforward and without any surprises. For example: the set of events is a topological Boolean algebra if, and only if the universe of situations is a (non-ordinary) topological Boolean algebra. In order to do this it is necessary for the reader to engage in tiring formalism. We plan, therefore, a thorough description of the language of the theory of reification. It will be open language, that is to say, without bound variables and quantifiers, and furthermore, as simple as possible. The simplification of the language, however, entails certain complications, in that statements, such as the one above about the transfer of properties

of the universe of situations to events, must be expressed after all in rather simple words as metatheorems about the inferential equivalence of sets of formulas.

As already mentioned above, the theory of reification is based on non-Fregean logic. The logical axioms and inference rules of this logic will be discussed shortly.

I use, of course, nations of set theory. It occurs in metatheory in its normal role. On the other hand, in the verbal text of reification theory I use 'naive' nations of a set (class), relation and function. For example, (1) reification is mapping (function)

$$\mathbf{T} : ST \to Ob$$

of the set (universe) of situations to the set (universe) of objects, (2) the set of events is a $\mathbf{T}$-image of the set of all situations, (3) the set of events contains the subset of positive events, which distinguish the primary predicate P, introduced below.

2.

The language J of the theory of reification contains sentence variables p, $q, r, \ldots$, and name variables $x, y, z, \ldots$ Let us also give them numerical indices such as p_k and x_k for $k = 1, 2, \ldots$ Accordingly, in language J we have two types of formulas: sentence formulas and name formulas. In J we distinguish two sublanguages. The first, J_1, is specified by an alphabet containing besides variables (p_k, x_k) a name constant l, a two-place functor f, a two-place predicate H, a predicate of identity $\equiv$, truth functional sentence connectives $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$ (negation, conjunction, disjunction, implication, and material equivalence), and an identity connective, noted likewise by the symbol $\equiv$. Therefore, if η, ϑ are name formulas (in particular name variables or the name constant), and α, β are sentence formulas (in particular sentence variables), the expression $f(\eta, \vartheta)$ is a name formula, and the expressions $H(\eta, \vartheta)$, $\eta \equiv \vartheta$, $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \Rightarrow \beta$, $\alpha \Leftrightarrow \beta$ as well as $\alpha \equiv \beta$ are sentence formulas.

Instead of the formula $1 = 1$, we write $\mathbf{1}$. We write $\phi \not\equiv \Psi$ instead of $\neg(\phi \equiv \Psi)$.

The difficulties which the idea of situation encounters are concentrated around sentence-creating formators, which are symbols of functions to the universe of situations. Such a predicate H is the symbol of

the operation

$$H : x, y | \rightarrow H(x, y),$$

transforming objects x, y into the situation $H(x, y)$ which occurs (is a fact), or does not.

The same concerns the predicate of identity. In a similar way, the negation operator $\neg$ is the symbol of the operation

$$\neg : p | \rightarrow \neg p,$$

transforming the situation p into the situation $\neg p$; here it is such that among two situations p and $\neg p$ precisely one occurs (is a fact). We treat the other truth-functional connectives and the identity connective analogously.

And so, if we want to get rid of situations in favor of events, we should also replace the operations given by sentence-forming formulas with operations given by proper name-forming formulas (functors, symbols of operations transforming objects into objects). Reification also has to characterize the newly added operations as isomorphic with the operations on the set of situations which we intend to get rid of. In this connection there exists in language J not only a one-place formator of reification $\mathbf{T}$, but also the functors $-$, $\cap$, $\cup$, $\doteq$, $\div$ corresponding to the truth-functional connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, and further a functor $\bigcirc$ corresponding to both the identity connective and the identity predicate, as well as a two-place functor h corresponding to predicate H. Moreover, we also add to language J a one-place predicate P, which corresponds to assertion, or precisely, to the assertion connective (to be a fact), provided that one would add '*explicate*' to the alphabet of J.

The second sublanguage, J_2, is marked by an alphabet, which apart from name variables x_k contains the constant 1, functor f, the newly added functors $-$, $\cap$, $\cup$, $\doteq$, $\div$, $\bigcirc$, h, as well as the predicate of identity and truth-functional connectives.

3.

In language J – and therefore in sublanguages J_1 and J_2 – an (absolute) non-Fregean logic[2] is assumed; that is to say, the consequence relation Cn, designated by the below mentioned logical axioms and two rules of inference: modus ponens and the substitution of sentence and name formulas by variables p_k and x_k, respectively. The logical axioms comprise (1) the well-known axioms for truth-functional connectives,

(2) formulas $x \equiv x$, $p \equiv p$, and $(p \equiv p) \Rightarrow (p \Rightarrow q)$, as well as (3) invariance axioms for each (!) formator in J individually. For example, I give only the most simple of them:

$$(p \equiv q) \Rightarrow (\neg p \equiv \neg q), \qquad (x \equiv y) \Rightarrow (-x \equiv -y)$$
$$(p \equiv q) \Rightarrow \big(\mathbf{T}(p) \equiv \mathbf{T}(q)\big), \qquad (x \equiv y) \Rightarrow p(y)\big).$$

The set of tautologies = $Cn(\emptyset)$, where $\emptyset$ is the null set. Examples of tautologies (aside from the logical axioms):

$$p \vee \neg p, \qquad p \neq \neg p, \qquad p \Leftrightarrow \neg\neg p,$$
$$(p \equiv q) \Leftrightarrow (q \equiv p), \qquad (x \equiv y) \Leftrightarrow (y \equiv x).$$

The equivalence of the set of sentence formulas X, Y implies that $Cn\ (X) = Cn\ (Y)$. The equivalence of sets X, Y on the basis of A implies that $Cn\ (A + X) = Cn\ (A + Y)$. The idea of equivalence (inferential) also applies to individual formulas (unit sets).

The reader will notice that the limitation of consequent Cn to sublanguage J_2 is Fregean logic in that language.

The set of axioms Ar of the theory of reification takes in, apart from the two formulas

(I) $\quad (\mathbf{T}(p) \equiv \mathbf{T}(q)) \equiv (p \equiv q),$

(II) $\quad (P(\mathbf{T}(p)) \equiv p,$

the following group of formulas

(III) (a) $\quad \mathbf{T}(1) \equiv 1$

(b) $\quad \mathbf{T}(x \equiv y) \equiv (x \bigcirc y)$

(c) $\quad \mathbf{T}\ (H(x,y)) \equiv h(x,y)$

(d) $\quad \mathbf{T}(\neg p) \equiv -\mathbf{T}(p)$

(e) $\quad \mathbf{T}(p \wedge q) \equiv \big(\mathbf{T}(p) \cap \mathbf{T}(q)\big)$

(f) $\quad \mathbf{T}(p \vee q) \equiv \big(\mathbf{T}(p) \cup \mathbf{T}(q)\big)$

(g) $\quad \mathbf{T}(p \Rightarrow q) \equiv \big(\mathbf{T}(p) \doteq \mathbf{T}(q)\big)$

(h) $\quad \mathbf{T}(p \Leftrightarrow q) \equiv \big(\mathbf{T}(p) \div \mathbf{T}(q)\big).$

In this group we do not have the axioms corresponding to the identity connective:

(1) $\quad \mathbf{T}(p \equiv q) \equiv \big(\mathbf{T}(p) \bigcirc \mathbf{T}(q)\big).$

The equality, however, results from (I) and (IIIb):

$$\mathbf{T}(p \equiv q) \equiv \mathbf{T}\left(\mathbf{T}(p) \equiv \mathbf{T}(q)\right) \equiv \left(\mathbf{T}(p) \bigcirc \mathbf{T}(q)\right).$$

Further simple conclusions from our set of axioms:

(2) $\mathbf{T}\left(P\left(\mathbf{T}(p)\right)\right) \equiv \mathbf{T}(p)$.

(3) $P\left(\mathbf{T}\left(P\left(\mathbf{T}(p)\right)\right)\right) \equiv P\left(\mathbf{T}(p)\right)$,

(4) $P(x \bigcirc y) \equiv (x \equiv y)$.

We introduce two other symbols in language J: a one-place sentence connective $\Box$, and, corresponding to it, a one-place functor I. The first is counted in the alphabet of language J_1, and the second in language J_2. These symbols are defined as follows:

(5) $\Box p \equiv (p \equiv 1)$, $\quad Ix \equiv (x \bigcirc 1)$.

It, therefore, immediately follows that

(6) $\mathbf{T}(\Box p) \equiv I\left(r(p)\right)$.

4.

Events are the reification of situations. The idea of event can be defined using quantifiers:

$$x \text{ is an event } = df\, \exists p(x \equiv \mathbf{T}(p)).$$

Such a definition exceeds the means of open language J. It is not, however, indispensable to us here. In the domain in which it will occur here, the use of the term "event" or "set of events" is quite correct. We will say, for example, that 1 is an event because it is the reification of situation 1. We have at the very least two events, 1 and -1, since $1 \equiv \mathbf{T}(1) \neq \mathbf{T}(\neg 1) \equiv -\mathbf{T}(1) \equiv -1$. The set of events is a certain algebra, i.e. possess a certain algebraic structure with respect to the operations $-$, $\cap$, $\cup$, $\doteq$, $\div$, $\bigcirc$, since these operations do not lead out of the set of events. For we have: $x = \mathbf{T}(p) \Rightarrow -x = \mathbf{T}(\neg p)$, etc. From this point of view, the equalities in (III) are partial definitions (since they only concern events) of the operations mentioned on the right-hand side, and axioms (II) also defines predicate p, partially, but precisely enough in order to be able and event -1 is not.

Formula $\alpha[p_1, \ldots, p_k]$ is a condition imposed on situations. When this formula generally occurs, it expresses a certain property of a general

situation and of the operations whose symbols occur in the formula. If, for example, in general:

$$(p \equiv q) \equiv (q \equiv p),$$

we would say that the operation of identification is commutative in the set of situations.

Similarly, for formulas $\beta[x_1, \ldots x_k]$ with name variables. A formula of this type expresses a certain property of the set of all objects as well as operations which comprise the symbols. One can, so to speak, complete such a formula and obtain conditions restricted to events. A condition of this sort has the form (**T**-normal form):

$$x_1 \equiv \mathbf{T}(p_1) \wedge \ldots \wedge x_k \equiv \mathbf{T}(p_k) \Rightarrow \beta[x_1, \ldots, x_k].$$

An equivalent variation is the formula

$$\beta[\mathbf{T}(p_1, \ldots, \mathbf{T}(p_k)],$$

or in other words, the result of substituting $\mathbf{T}(p_n)$ for x_n in

$$\beta[x_1, \ldots, x_k] \qquad \text{for } n = 1, \ldots, k.$$

In particular the commutativity of the operation of identification of events – and there are two such operations – is expressed in the formulas:

$$x \equiv \mathbf{T}(p) \wedge y \equiv \mathbf{T}(q) \Rightarrow (x \equiv y) \equiv (y \equiv x),$$

$$x \equiv \mathbf{T}(p) \wedge y \equiv \mathbf{T}(q) \Rightarrow (x \bigcirc y) \equiv (y \bigcirc x),$$

$$(\mathbf{T}(p) \equiv \mathbf{T}(q)) \equiv (\mathbf{T}(q) \equiv \mathbf{T}(p))$$

$$(\mathbf{T}(p) \bigcirc \mathbf{T}(q)) \equiv (\mathbf{T}(q) \bigcirc \mathbf{T}(p)) .$$

Now if α expresses a certain property of situations and β expresses a certain property of events, we cannot employ a formula of the sort $\alpha \Leftrightarrow \beta$ for a statement of the fact that situations have the first property if, and only if, events have the second property. We can state such a fact in the metalanguage saying that α and β are equivalent on the grounds of the set of axioms of the theory of reification. It is clear that, having quantifiers in J, we would be able to use the formulas

$$\forall\alpha \Leftrightarrow \forall\beta \text{ and even } \forall\alpha \equiv \forall\beta,$$

where the left and right-hand sides are the so-called closure of formulas α and β, respectively.

5.

It is easy to verify that the following formulas

$$p \equiv 1 \wedge p \equiv \neg 1, \quad \mathbf{T}(p) \equiv 1 \wedge \mathbf{T}(p) \equiv -1,$$

of which the second is equivalent to the r-normal formula

$$x \equiv \mathbf{T}(p) \Rightarrow (x \equiv 1 \wedge x \equiv -1),$$

are equivalent on the basis of axiom scheme (I). Loosely speaking, exactly two situations exist if, and only if, exactly two events exist. Formally, I do not see anything deep in this, but I leave the reader to reflect on the naturalness of the Fregean axiom (only two situations exist), as well as on the assumption of the two-element set of events.

If we accept the Fregean axiom, then the identity connective and the material equivalence connective are completely indistinguishable. Enriched by the Fregean axiom, the theory of reification becomes an open theory based on Fregean logic (the calculus of truth-functional connectives), and the proof of its consistency is a very easy task. This being so

> the theory of reification (without the Fregean axiom) is a consistent theory

Later, we will consider certain natural assumptions about the universe of situations which result from the Fregean axiom, and are indeed weaker than it. At present we ask what the "appearance" of the set of events is, if we do not make any extra-logical assumptions concerning situations. Firstly, the set of events is an algebra with respect to the operators $-$, $\cap$, $\cup$, $\doteq$, $\div$, and $\bigcirc$, in which a certain element l is designated. In the set of events, the set of positive events is also designated. The algebra of events with a designated subset of positive thus determines a certain 'logical matrix', in which the operators and designated subset are connected in the following dependencies. For any event x, y:

(7) $P(-x) \equiv \neg P(x)$,

(8) $P(x \cap y) \equiv P(x) \wedge P(y)$,

(9) $P(x \cup y) \equiv P(x) \vee P(y)$,

(10) $P(x \doteq y) \equiv \neg P(x) \vee P(y)$,

(11) $P(x \div y) \equiv (p(x) \wedge P(y)) \vee (\neg P(x) \wedge \neg P(y))$,

(12) $P(x \bigcirc y) \equiv (x \equiv y)$.

Changing the identity connective to the material equivalence connective in the above equations, we obtain a characterization of the matrix of events of the so-called SCI-model, or of the model of the classical calculus of sentences with the identity connective.[3]

This is virtually all on the theme of events which flows from pure logic because the information that $1 \bigcirc 1 = 1$ is a consequence of the conventional designation of the situation $1 \equiv 1$. Similarly, the equality $P(Ix) \equiv (x \equiv I)$ is an immediate result from the definition of functor I.

6.

Reification is an isomorphic image of the universe of situations to the algebra of events. The theory of reification, however, provides the tools with the help of which we replace the theory of situations with the theory of events. For every sentence formula α of language J_1, we define a name formula α_T, such that the following formulas will be statements of the theory of reification:

(13) $\alpha_T \equiv \mathbf{T}(\alpha)$

(14) $P(\alpha_T \equiv \alpha$

The definition of name formulas $\mathcal{L}_T$ is inductive:[4]

$$[p_k]_T = \mathbf{T}(p_k), \qquad 1_T = 1,$$

$$H(H, \vartheta)_T = h(\eta, \vartheta), [\eta \equiv \vartheta]_T = (\eta \bigcirc \vartheta)$$

$$[\Box\alpha]_T = I(\alpha_T), \qquad [\neg\alpha]_T = -(\alpha_T),$$

$$[\alpha \,\S\, \beta]_T = (\alpha_T \,\%\, \beta_T)$$

where § is a two-place truth-functional connective or the identity connective, and % is a two-place functor ($\cap$, $\cup$, $\doteq$, $\div$, $\bigcirc$) corresponding to the connective §. Elementary inductive reasoning proves (13) and (14).

Let α, or in the more suggestive notation:

(α) $\alpha[p_1, \ldots p_k, x, y, z]$

be a sentence formula in J_1 with marked variables. In order to avoid confusion with variables above, we assume that the name variables x, y, z are different from $x_1, \ldots, x_k$. We note the name formula α_T as follows:

$$(\alpha_T) \qquad \alpha_T[\mathbf{T}(p_1, \ldots, \mathbf{T}(p_k), x, y, z].$$

It originates from α by the "mechanical inscription" of $\mathbf{T}(p_i)$, l, h, $\bigcirc$, I, $-$, $\cap$, $\cup$, $\dot{=}$, $\div$, $\bigcirc$ instead of p_i, l, H, $\equiv$, $\Box$, $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\equiv$. In compliance with (1) and (2), we now have:

$$(15) \qquad \alpha_T[\mathbf{T}(p_1), \ldots, \mathbf{T}(p_k), x, y, z] \equiv \mathbf{T}(\alpha[p_1, \ldots, p_k, x, y, z]),$$

$$(16) \qquad P(\alpha_T[\mathbf{T}(p_1), \ldots, \mathbf{T}(p_k), x, y, z]) \equiv \alpha[p_1, \ldots, p_k, x, y, z].$$

Name formula α does not belong to the language J_2. If, however, we replace $\mathbf{T}(p_i)$ by x_i in it, we obtain the name formula polynomial α_T^* in J_2, which we also note as:

$$(\alpha_T^*) \qquad (\alpha_T^*)[x_1, \ldots, x_k, x, y, z].$$

From the construction one can see that if $x_1, \ldots, x_k$ are events, then α_T^* is also an event, and we have the statement: if $x_i = \mathbf{T}(p_1)$ for $i = 1, \ldots, k$, then

$$(17) \qquad \alpha_T^*[x_1, \ldots, x_k, x, y, z] \equiv \mathbf{T}(\alpha[p_1, \ldots, p_k, x, y, z]).$$

Moreover, the following 'metatheorem 1' holds:
Sentence formulas

$$\alpha[p_1, \ldots, p_k, x, y, z],$$
$$P(\alpha_T)p_1, \ldots \mathbf{T}(p_k), x, y, z]),$$
$$x_1 \equiv \mathbf{T}(p_1) \wedge \ldots \wedge x_k$$
$$\equiv \mathbf{T}(p_k) \Rightarrow P(\alpha_T^*[x_1, \ldots, x_k, x, y, z]),$$

are equivalent on the grounds of the theory of reification.

Name formulas α_T and α_T^*, of which only the second belongs to language J_1, are functions of the sentence formula α of language J_1. Two sentence formulas are also functions of sentence formula α, that is to say, $\alpha[p_1, \ldots, p_k, x, y, z]$,

$$P(\alpha_T) \text{ or } P(\alpha_T[\mathbf{T}(p_1), \ldots, (p_k), x, y, z)$$

and

$$P(\alpha_T^*) \text{ or } P(\alpha_T^*[x_1, \ldots, x_k, x, y, z]).$$

The second sentence formula, $P(\alpha^*_T)$ is called an $\mathbf{T}$-translation of sentence formula α. According to our metatheorem:

> α holds for every situation $p_1, \ldots, p_k$ (and objects x, y, z) if, and only if, $P(\alpha^*_\tau)$ holds for every event $x_1, \ldots, x_k$ (and objects x, y, z).

The translation $P(\alpha^*_\tau)$ of formula α in J_1 contains the predicate P, and that is why it does not belong J_2. It results, however, that in the case of certain sentence formulas in J_1 one can eliminate the predicate P from their $\mathbf{T}$-translations. For such sentence formulas in J_1, their $\mathbf{T}$-translations are sentence formulas in J_2.

We define the class of sentence formulas in J called truth-functional polynomials in equality, or PR-polynomials for short. Every equality $\phi \equiv \Psi$ as well as formulas defined as equality, such as $\Box\alpha$, are polynomial.

Further PR-polynomials are constructed with the help of truth-functional connectors from simple PR-polynomials. If, therefore, α, are PR-polynomials, the following are also PR-polynomials: $\alpha, \alpha, \alpha, \alpha, \alpha$.

Since $P(x \bigcirc y) \equiv (x \equiv y)$, for established name formulas η, ϑ and sentence formulas α, β:

$$(18) \qquad P([\eta \equiv \vartheta]_T \equiv P(\eta \bigcirc \vartheta) \equiv (\eta \equiv \vartheta),$$

$$(19) \qquad P([\alpha \equiv \beta]_T) \quad P(\alpha^* \bigcirc \beta^*_T) \quad (\alpha^*_T \equiv \beta^*_T).$$

$$(20) \qquad P([\Box\alpha]^*_T) \quad P\left(I(\alpha^*_T)\right) \equiv P(\alpha^*_T \bigcirc 1) \quad (\alpha^*_T \equiv 1).$$

If the expressions $\eta \equiv \vartheta$, $\alpha \equiv \beta$, $\Box\alpha$ are formulas in J_1, then their $\mathbf{T}$-translations stand on the left-hand side in (18), (19), and (20), and on the right-hand side we have certain equalities belonging to J_2. We call them proper $\mathbf{T}$-translations of corresponding formulas of J_1. They are equalities among certain name (polynomials) formulas in J_2.

Applying the given equalities in Section 5, we find the proper $\mathbf{T}$-translation of any PR-polynomial in J_1. They will be PR-polynomial in J_2. More precisely, if γ is a PR-polynomial in J_1, then its proper $\mathbf{T}$-translation in J_2 is analogous to the PR-polynomial $\bar{\gamma}$, which originates from γ by "mechanical inscription" of their proper $\mathbf{T}$-translations instead of equality.

The equality in J_1:

(21) $(x \bigcirc y) \qquad I(x \bigcirc y)$,

is a proper $\mathbf{T}$-translation of both equalities in J_2:

(22) $(p \equiv q) \equiv \Box(p \equiv q), \quad (x \equiv y) \equiv \Box(x \equiv y)$.

Similarly, the equality in J_1:

(23) $-(x \bigcirc y) \qquad I\,(-(x \bigcirc y))$,

is a common proper $\mathbf{T}$-translation of two equalities in J_2:

(24) $(p \not\equiv q) \equiv Box(p \not\equiv q), \quad (x \not\equiv y) \equiv \Box(x \not\equiv y)$.

The proper $\mathbf{T}$-translations of PR-polynomials in J_1:

(25) $p \equiv 1 \vee p \equiv \neg 1$,

(26) $(p \equiv q) \equiv 1 \vee (p \equiv q) \equiv \neg 1$,

(27) $(x \equiv y) \equiv 1 \vee (x \equiv y) \equiv \neg 1$,

are the following PR-polynomials in J_2:

(28) $x \equiv 1 \vee x \equiv -1$

(29) $(x \bigcirc y) \equiv 1 \vee (x \bigcirc y) \equiv -1$ ($\mathbf{T}$-translation for (22) and (23)).

An identical relationship occurs for proper $\mathbf{T}$-translations as for ordinary $\mathbf{T}$-translations:

PR-polynomial $\gamma(p_1, \ldots, p_k, x, y, z)$ holds for every situation $p_1, \ldots, p_k$ (and objects x, y, z) if, and only if, its proper $\mathbf{T}$-translation $\bar{\gamma}(x_1, \ldots, x_k, x, y, z)$ holds for every event $x_1, \ldots, x_k$ (and object x, y, z).

7.

Let us consider the following equalities in J_1

(30) $p \wedge q \equiv q \wedge p, \qquad p \vee q \equiv q \vee p$,

(31) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r, \qquad p \vee (q \vee r) \equiv (p \vee q) \vee r$,

(32) $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$,

(33) $(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r)),$

(34) $(p \wedge l) \equiv p, \qquad (p \vee \neg 1) \equiv p,$

(35) $(p \vee \neg 1) \equiv p, \qquad (p \wedge \neg p) \equiv \neg l,$

(36) $(p \Rightarrow q) \equiv (\neg p \vee q), \qquad (p \Leftrightarrow q) \equiv (p \Leftrightarrow q) \wedge (q \Rightarrow p),$

(37) $\Box\Box p \equiv p, \qquad \Box(\Box p \Rightarrow p),$

(38) $\Box(p \wedge q) \equiv \Box p \wedge \Box q,$

(39) $(p \equiv q) \equiv \Box(p \Leftrightarrow q).$

The proper **T**-translations of these formulas are these equalities in J_2:

(40) $x \cap y \equiv y \cap x, \qquad x \cup y \equiv y \cup x,$

(41) $x \cap (y \cap z) \equiv (x \cap y) \cap z, \qquad x \cup (y \cup z) \equiv (x \cup y) \cup z,$

(42) $x \cap (y \cup z) \equiv (x \cap y) \cup (x \cap z),$

(43) $x \cup (y \cap z) \equiv (x \cup y) \cap (x \cup z),$

(44) $x \cap l \equiv x, \qquad x \cup -l \equiv x,$

(45) $x \cup -x \equiv l, \qquad x \cap -x \equiv -1,$

(46) $x \dot{=} y \equiv -x \cup y, \qquad x \div y \equiv (x \dot{=} y) \cap (y \dot{=} x),$

(47) $IIx \equiv Ix, \qquad (Ix \dot{=} x) \equiv l,$

(48) $I(x \cap y) \equiv Ix \cap Iy,$

(49) $x \bigcirc y \equiv I(x \div y).$

We notice that from (37) we have $(\Box l \equiv l) \equiv (l \equiv l)$, and therefore $\Box l \equiv l$. Similarly, $I1 \equiv 1$, since $1 \bigcirc 1 \equiv 1$.

Let $\gamma(p, q, r)$ and $\delta(p, q)$ respectively be conjunctions of equalities (30), ..., (36), and (37), (38), (39). Proper **T**-translations of these conjunctions $\bar{\gamma}(x, y, z), \bar{\delta}(x, y, z)$ are respectively conjunctions of equali-

ties (40), ... (46) and (47), (48), (49).

From earlier considerations, equivalence results on the basis of the theory of reification of formulas $\gamma(p, q, r)$ and $x \equiv \mathbf{T}(p) \wedge y \equiv \mathbf{T}(q) \wedge z \equiv \mathbf{T}(r) \Rightarrow \bar{\gamma}(x, y, z)$, and formulas $\gamma(p, q, r) \wedge \delta(p, q)$ and $x \equiv r(p) \wedge y \equiv r(q) \wedge z \equiv r(r) \Rightarrow \bar{\gamma}(x, y, z) \wedge \bar{\delta}(x, y, z)$. Equivalence means that:

1. the algebra of events is a Boolean algebra if, and only if, the universe of situations is an (unusual) Boolean algebra.

2. the algebra of events is a topological Boolean algebra if, and only if, the universe of situations is an (unusual) topological Boolean algebra.

The addition of the term "unusual" informs us that one can talk about the elements of the algebra of situations with the help of formula with sentence variables. Its unusualness manifests itself among other things in the following circumstance. We acknowledge the sentence formula $p \vee \neg p$ stating that $p \vee \neg p$. The situation, that $p \vee \neg p$, is a unity in the unusual Boolean algebra of situations. Its corresponding event $\mathbf{T}(p) \cup -\gamma(p)$ is also a unit in the Boolean algebra of events. The formula $\mathbf{T}(p) \cup -\mathbf{T}(p)$ is, however, a name formula and its acceptance in the ordinary sense is excluded.

8.

At present, we strengthen the theory of reification in accepting as an added axiom the conjunction considered above

(IV) $\quad \gamma[p, q, \mathbf{T}] \wedge \delta[p, q]$,

and therefore the assumption that situations create an (unusual) topological Boolean algebra. The result is that the theory of situations with axiom scheme (IV) (based on sentential logic with the identity connective) is identical to system S_4 of Lewis.[5] On the basis of the theory of reification we now come to the conclusion that the algebra of events is a topological Boolean algebra, and that positive events are ultrafilters in this algebra.

That events determine the Boolean algebra is the desired result. In the foundations of a probability calculus it usually assumed that events constitute α Boolean algebra. It is not at all assumed, however, that there is a defined topological interior operator I in the algebra of events. One can suppose that operator I is omitted since it is trivial, i.e. $Ix = x$ for every event x.

It happens, however, that the interior operator is essential in the Boolean algebra of events as reifications of situations. We will demonstrate that operator I is trivial in the algebra of events when there exists precisely two events.

For we have the following metatheorem 2:

Formulas $x \equiv \mathbf{T}(p) \Rightarrow Ix \equiv x$ and $z \equiv \mathbf{T}(r) \equiv (z \equiv 1) \vee (z \equiv O)$, where $O \equiv -1$, are equivalent on the basis of the (strengthened) theory of reification.

Proof: From the equality (49) and the Boolean equality $((-z) \div y) \equiv (z \div (-y))$ we obtain the equality $(-z \bigcirc 1) \equiv (z \bigcirc -1)$. We assume that if x is an event, then $Ix \equiv x$ or $(x \bigcirc 1) \equiv x$. Let z be an event, then: $-z \bigcirc 1 \equiv -z$, also $-(z\ bigcircl) \equiv -z$. Consequently, $(-z \bigcirc 1) \equiv -(z \bigcirc 1)$, and therefore $(z \bigcirc -1) \equiv -(z \bigcirc 1)$. Further, $P(z \bigcirc -1) \equiv \neg P(z \bigcirc 1)$, and hence $(z \equiv -1) \equiv (z \not\equiv 1)$. And so $(z \equiv 1) \vee (z \equiv -1)$. The proof in the other direction is immediate.

In order to explain the fact of the notorious omission of the operator I in the Boolean algebra of events, one can assume that this operator is not indeed trivial, but is so particular that its role is of little importance and one can understand it all the time. Operator I is a special operator of this sort such that for every event x:

$$(Ix \equiv 1) \vee (Ix \equiv 0),$$

or equivalently: $x \not\equiv 1 \Rightarrow Ix \equiv 0$. This sort of operator I is often called the Henle operator in Boolean algebra, and Henle algebra is a Boolean algebra with such an operator.

The supposition that the algebra of events (reifications of situations) is a Henle algebra seems quite sound against the background of the above considerations. The goal of those considerations was the demonstration that designated elements 1 and 0 of Boolean algebra are designated in the Henle algebra of events also as determined events. Below, we abandon in many places – for simplicity – the forms of the antecedents in implication $x \equiv \mathbf{T}(p), \ldots, z \equiv \mathbf{T}(r)$; they say that $x, \ldots, z$ are events.

9.

Open elements in the topological Boolean algebra of events are those events x such that $Ix \equiv x$. It is worth noticing that the equality so written is a proper r-translation of the equality $\Box p \equiv p$. Open events merit, therefore, the name of "determined" events. The are those events in which the operator I is trivial. It is clear that events 1 and 0 are open. It is easy to notice that the algebra of events is a Henle algebra if, and only if, 1 and 0 are the sole open events (determined), or in other words, if for any event z this formula occurs:

(50) $z \equiv Iz \Rightarrow (z \equiv 1) \vee (z \equiv 0).$

If x, y are events, the event $x \bigcirc y$ is open, for we have $x \bigcirc y \equiv I(x \div y) \equiv II(x \div y) \equiv I(x \bigcirc y)$. On the other hand, if z is an open event, then $z \equiv (z \bigcirc 1)$. Hence, it results simply that the algebra of events is a Henle algebra if, and only if, 1 and 0 are its sole open elements, or that the alternative

(51) $(x \bigcirc y \equiv 1) \vee (x \bigcirc y \equiv 0)$

occurs for any event x, y.

Similarly, as in the case of each topological Boolean algebra, so now

$$(Ix \equiv x) \wedge (Iy \equiv y) \Rightarrow I(x \,\S\, y) \equiv x \,\S\, y,$$

where ?? is one of the two functors $\cap$, $\cup$. It results, therefore, that an open event is a Boolean algebra if, and only if, the complement of an open event is also an open event:

(52) $Iz \equiv\Rightarrow I(-z) \equiv -z.$

One should underline here that formula (52) is equivalent to the equality at the base of the theory of reification

(53) $-Ix \equiv I(-Ix),$

which is a proper r-translation of the equality

(54) $\neg\Box p \equiv \Box\neg\Box p,$

characteristic of system S_5 of Lewis.

Proof of the equivalence of formulas (52) and (53): We assume (53) and suppose that $Iz \equiv z$. Then $-z \equiv -Iz \equiv I(-Iz) \equiv I(-z)$. We now assume (52). Since $IIx \equiv Ix$, therefore $I(-Ix) \equiv -Ix$.

The assumption that open (determined) events constitute a Boolean algebra is very attractive. The realization that such an assumption is

equivalent to the supposition that the algebra of events is a Henle algebra is interesting. In other words, open (determined) events constitute a Boolean algebra if, and only if, the algebra of determined events consists only of 0 and 1.

We have metatheorem 3:

Formulas (50) and (52) are equivalent on the basis (strengthened) of the theory of reification.

Proof: Accepting (50), we assume that $Iz \equiv z$. Then $I(-z) \equiv -z$ both when $z \equiv 1$ and $z \equiv 0$. We now accept (52) and suppose that $Iz \equiv z$. We have $I(-z) \equiv -z$, and therefore $I(-z) \equiv -Iz$ or $(-z \bigcirc 1) \equiv -(z \bigcirc 1)$. And so $P(-z \bigcirc 1) \equiv \neg p(z \bigcirc 1)$ and further $(-z \equiv 1) \equiv (z \not\equiv 1)$. If therefore $z \not\equiv 1$, then $-z \equiv 1$, or $z \equiv 0$.

If the algebra of events is a Henle algebra, then the alternative formula (52) holds for events. Hence it follows – compare with the end of the sixth section – that for any situation p, q:

$$(55) \qquad ((p \equiv q) \equiv 1) \vee ((p \equiv q) \equiv 0),$$

where instead of $\neg 1$ we write 0. Conversely, from (55) it follows that (51) holds for every event.

Generalizing somewhat, we assume that disjunction (51) holds for every object x, y. Then also for every object x, y:

$$(56) \qquad ((x \equiv y) \equiv 1) \vee ((x\ equivy) \equiv 0.$$

Conversely, if (56) holds for every object x, y then alternative formula (51) also generally holds.

Formulas (55) and (56) determine together the so-called ontological principle of the zero-one operation of identification (situations, objects) which was assumed in the foundation of Wittgenstein's formalized ontology.

10.

The theory of reification is a theory in which we speak of situations and objects, and particularly of events as reifications of situations. It is based in an essential way on non-Fregean logic with an identity connective. The fundamental axioms as well as the later accepted axioms of the theory of reification are equivalent. Theories based on an equivalent axiom scheme possess special properties in non-Fregean logic. And the

strengthened theory of reification possesses the following property: if α is a theorem, then $\Box\alpha$ is also a theorem.

But another matter is more important and more difficult. The theory of reification is an ontic theory (not metatheoretical). It not only differentiates and identifies situations, as it employs an identity connective, but also employs a formator of the category name/sentence which – like the category of Słupecki's star – is unknown in the standard formalized theory of modern logic. This very formator, axiomatically characterized, permits us to speak about the one-to-one mapping of the structure of the universe of situations onto the algebra of events as well as about a certain parallelism of properties of the universe of situations and the algebra of events.

From the theory of situations we rather mechanically read the existence of certain translations (**T**-translation, proper **T**-translation) of sentence formulas with variables $p, q, r, \ldots$ into formulas in which these variables do not occur. The existence of r-translations is noticed and described by M. C. Cresswell in 'Functions of Propositions', *The Journal of Symbolic Logic* **31**, 1966, 545–560. The result as well as the entire construction of Cresswell is contained completely in the metatheory (well-formed, and also semantical in the current sense), in which two separate languages are investigated, corresponding more or less to our sublanguages J_1 and J_2; but one cannot forget that the predicate P is also in J_2. Cresswell's translation corresponds to our metatheorem 1 and transforms tautologies of J_1 into theorem of theory in J_2, whose axioms are formulas of the type (7), ..., (12). From Cresswell's final considerations one can guess how to pass over from r-translations to proper r-translations.

The fact that Cresswell uses the term "proposition" instead of "situation" does not have any influence on his formal work. The term "proposition" – like the opposition between extensional and intensional – stands at the center of a serious confusion of understanding in Anglo-American philosophy. A certain settlement of the basic semantical terminology is necessary for the needs of non-Fregean logic. But about that some other time. In the case of Cresswell what is essential is that his construction (1) is exclusively metatheoretical, and 92) joins – in his mind – the opinion of many logicians "that all benefits of FC (the non-Fregean sentential logic) can be obtained within the ordinary predicate calculus"; on the other hand "ordinary predicate calculus" is Fregean logic (and theory) in a language with only name (and not sentence)

variables.

If it is interesting to the reader, he can continue the semantic analysis of the expression "all benefits". I believe that reflection rather on the general-methodological circumstances is more fertile. And so Fregean logic, and the theories added to its foundations with set theory in the forefront determines the mental paradigm of modern logic. Even formal investigations in so-called intensional logic with the help of the method of indices, principally in the California school (Dana Scott, Richard Montague, and others) bring about only a certain 'loosening' of the extensional Fregean paradigm. Current reflection about objects and their properties and relationships is completely contained, in its framework but though this paradigm does not exclude sentences and statements (that would be absurd), i.e. expressions describing situations and stating facts – there is not any place in it for the non-trivial theory of situations, since the axiom of Frege reduces the number of situations to two: 1 and 0.

Let us notice that "the holding good" of the Fregean paradigm is rather closely connected with the tendency derived from neo-positivism to transfer logical problems to metatheory. In itself this tendency is not bad. Its fruit is, after all, such a beautiful theory as the (Fregean) theory of models. From this tendency, Cresswell's work is also derived, which as mentioned: (1) has a metatheoretical character, and (2) manages by the construction of a suitable translation – the tendency, based on Fregean logic, to replace the theory of situations by a theory of events as certain objects.

Non-Fregean logic is a generalization of the Fregean paradigm that fully retains both the property of extensionality and the classical principle of the two-valuedness of logic, but which frees us from the Fregean answer to the question: how many situations are there? The tautological answer to this question in non-Fregean logic is: there exist at least two situations. One can already see that the center of the confrontation of non-Fregean logic with the Fregean paradigm lies on the ontological level, and not on the metatheoretical level. The issue is "how many situations are there", and "what is the structure of the universe of situations". The theory of reification transfers this issue to events and shows that from the Fregean axiom it follows that only two events exist, which is absurd from the point of view of common intuition, and also from the theoretical construction, like probability theory in which the algebra of events appears.

Cresswell's work demonstrates that the transfer of the problem concerning situations to evens, and in particular the removal of the absurd consequence of the Fregean axiom about the existence of only two events, does not indispensably demand the theory of reification, and can be carried out in the metatheory within the limits of the Fregean paradigm. Such a feature of Cresswell's result seems to me particularly interesting.

Another matter is the status of the translation of sentence formulas with sentence variables into formulas in which these variables do not occur, and the connection of this translation with the theory of reification. The latter is the theory of isomorphism of the universe of situations and algebra of events. Events were built up by abstraction to such abstract objects that we can say that isomorphism occurs. And so it is a complete universal truth that the isomorphism of certain structures automatically implies the existence of a translation between theories of these structures, and that the implied translation by the isomorphism possesses the best logical properties (for example, maintenance of the structure of deductions and proofs). The reciprocal connection is rather loose. Besides, it depends on the "quality" of the translation. Not entering into an analysis here of what I called the quality of translation, one can state that the existence of a translation between theories (particularly the behavior of theorems) does not entail an isomorphism or even something weaker, but only the expression of relationship of structure of creation about which we speak in the theories connected with translations. And that is why the translation exposed by Cresswell would have little value if it were not that it is in principle the precise translation of the theory of situations to the theory of objects, which is induced by the isomorphism of the universe of situations and the algebra of events, called reification here.

On the other hand, description and examination of this isomorphism is impossible without the use (!) of non-Fregean logic with an identity connective as well as certain formators omitted by the Fregean paradigm (Słupecki's star, the formator of reification **T**).

Attention. The algebras of events, as mentioned, are SCI-models. A certain formal resemblance also exists between reification and semantical homomorphisms of the language of the classical calculus of sentences with identity of SCI-matrices. There is, however, a difference between them which – as the reader himself notices – has a categorical character.

11.

Addition concerning quantification. Cresswell's translation also takes into account quantifiers. On this point the text of his work is difficult reading ("somewhat messy" – as he writes). On the other hand, the account of quantifiers in the theory of reification is rather clear and as a consequence gives such a translation as we have in Cresswell. As opposed to Cresswell we take into account the simpler case: a universal quantifier binding name variables. This does not change essential things because the idea of universality applies analogically to situations as to objects.

We strengthen, therefore, language J with the symbol $\forall$, which also counts itself among J_1 and J_2. We add to J_2 (and thus to J) a formator G binding one name variable and which from the name formula $\eta[x]$ creates the name formula $G\eta[x]$. We strengthen the third group of axioms of the theory of reification with the schema:

(IIIi) $\quad \mathbf{T}(\forall x\alpha[y]) \equiv G_x\mathbf{T}(\alpha[x]).$

And that is all. From (II) we obtain immediately

$$P\left(\mathbf{T}(\alpha[x])\right) \equiv \alpha[x]$$

$$P\left(\mathbf{T}(\forall x\alpha[x]0\right) \equiv \forall x\alpha[x],$$

and with the help of the axioms of invariance for $\forall$ we get:

$$\forall xP\left(\mathbf{T}(\alpha[x])\right) \equiv \forall x\alpha[x].$$

Similarly, in virtue of (IIIi):

$$P\left(\mathbf{T}(\forall x\alpha[x])\right) \equiv P\left(G_x\mathbf{T}(\alpha[x])\right),$$

for which in the end:

$$P\left(G_x\mathbf{T}(\alpha[x])\right) \equiv \forall xP\left(\mathbf{T}(\alpha[x])\right).$$

This is an equality which can be added to the group of equalities (7), ..., (12).

I will mention that in the strengthened theory of reification (the universe of situations determines a topological Boolean algebra) formator has the property of operator of generalized meet in the topological algebra of events if, and only if, the weak Barcan schema occurs for situations:

$$\forall x\Box\alpha[x] \Leftrightarrow \Box\forall x\alpha[x].$$

This is not an equality, and there are difficulties, with the proper r-translation of this schema which disappear in the case of strong Barcan

schema:

$$\forall x \Box \alpha[x] \equiv \Box \forall x \alpha[x].$$

NOTES

* Translated by Theodore Stażeski.

[1] *Studia Locica* **28**, 1971, 7–17.

[2] R. Suszko: 1971, 'Identity Connective and Modality', *Studia Logica* **27**, 7–41.

[3] S. L. Bloom, R. Suszko, 1972: 'Investigations into Sentential Calculus with Identity', *Notre Dame Journal of Formal Logic* **13**, 289–308.

[4] The definition belongs to the metatheory. That is why we use the identity sign = in it to distinguish it from the sign of identity, which occurs in language J. This distinction is not indispensable.

[5] Compare R. Suszko, W. Żandarowska: 1971, 'Systemy S_4 i S_5 Lewisa a spójnik identyczności' (Lewis Systems S_4 and S_5 the Identity Connective)', *Studia Logica* **29**, 169–177.

KAZIMIERZ TRZESICKI

INTUITIONISM AND INDETERMINISM (TENSE-LOGICAL CONSIDERATIONS)

There are some connections between philosophical views of intuitionists and indeterminists – philosophers rejecting the thesis of predeterminism, i.e. – roughly speaking – the thesis that if α is a fact, then at any moment in the past it was the fact that it would be the fact that α; or – what is closer to the thesis as formulated in our formal language – one of the two sentences:

It will be the fact that α.
It will be the fact that not$-\alpha$,

where α does not refer to a past fact was true (determined) at any moment in the past. The aim of the present paper is to describe some of these connections in terms of tense logic.

To avoid the predeterministic consequence of the principle of the excluded middle Aristotle questioned the applicability of it to predictions of future contingencies in the famous "sea-fight" passage of "*On Interpretation*". It is well known that the ancient and medieval view that predictions of future contingencies are "neither true nor false" provided the original stimulus for Łukasiewicz's many-valued logic. Łukasiewicz argues that to reject the argument from the logical principle of the excluded middle in favour of the thesis of determinism, besides "true" and "false", a third logical value is needed.[1]

The rejection of the logical principle of the excluded middle is not an aim but only a consequence of more deep convictions as well for indeterminists as for intuitionists.[2] For them the most important is the view that a sentence is (now) a theorem (is true, respectively) if it is (now) (in some sense) determined, and that there are some sentences which are not determined yet (not proved and not disproved sentences, sentences referring to future contingencies, respectively). The determination is related to time. By time, indeterminists mean physical time as defined by changes of real world. For intuitionists mathematical knowledge is a certain type of creation within time.[3] The laws of time

Jan Woleński (ed.), Philosophical Logic in Poland, 271–296.

do not have to determine everything. A creation is possible only if there are some not determined deeds and things. That there are such matters Aristotle himself is convinced, for otherwise "there would be no need to deliberate or to take trouble, on the supposition that if we should adpot a certain course, a certain result would follow, while, if we did not, the result would not follow" (*De Interpretatione*). Every theory (theorem) has its history and its future. The future is only determined to a certain degree. I think that tense-logical considerations of this paper can have some value to better understanding of intuitionism and can also help to illuminate problems having to do with intuitionistic mathematics. Besides these pure philosophical speculations formal results of our considerations are interesting for both intuitionism and tense logic.[4]

The intuitionistic logic seems to be appropriate to formalize the view that future contingencies are neither true nor false. The sentence:

It will be the fact that α or it will be the fact that not$-\alpha$.

is not a law of tense logic based on intuitionistic logic.[5]

1. SEMANTICS

A world $\mathfrak{W}$ consists of a non-empty set W of states of the world. The members of the set (the states of the world) are indexed by elements of a set T (by the moments of time at which they take place)

W_t is a state of a world $\mathfrak{W}$ at a moment t. A binary relation $R(\subseteq W \times W)$ of precedence (earlier – later) is a relation between states of a world. A world $\mathfrak{W}$ is an ordered 3-tuple: $<W, R, W_n>$ (or simpler: $<W, R, n>$). W_n is a designated member of W (the actual state of $\mathfrak{W}$).

We can define a binary relation $R(\subseteq T \times T)$ of temporal precedence (earlier – later) as a relation between elements of T as follows: t_1Rt_2 iff $W_{t_1}RW_{t_2}$. n (now) is a designated element of T. By a time $\mathfrak{I}$ we understand an ordered 3-tuple: $<T, R, n>$.

The state of world $\mathfrak{W}$ at t, W_t, consists of a non-empty class $\mathcal{S}_t$ (of sets of situations possible at t). By a situation we mean a semantic correlate of a sentence. Hence the set of situations can be conceived as a set of sentences. A fact is a realized (or, determined) situation. Thus facts are semantic correlates of true sentences. We are interested in situations which are determined in a state of world, i.e. we are interested in facts which form that state of world. A situation is determined in W_t if it is determined by each of the possible sets of situations of which W_t

is composed. A class of sets of situations possible in a certain W_t is dependent on a moment of time (or equivalently, a state of the world). The sets possible in W_t from the point of view of t_1 do not have to be the same as the sets possible from the point of view of t_2.

The sets of situations are ordered by the relation of set-theoretical inclusion. This relation is reflexive and transitive. Thus the state of a world $\mathfrak{W}$ at t can be conceived as an ordered pair:

$$W_t(= <\mathcal{S}_t, \leq>)$$

where $\mathcal{S}_t$ is a non-empty set (of sets of situations possible from the point of view of the actual state of world) and $\leq$ ($\subseteq \mathcal{S}_t \times \mathcal{S}_t$) is a reflexive and transitive relation.

We will describe temporal relations between facts in a language with tense operators. Thus we have to be able to talk about temporal relations between sets of situations. The situations which are semantic correlates of past-tense sentences (future-tense sentences) should be related to some situations in a certain past state of the world (future state of the world). Moreover, the situations in a certain past (future) state of the world ought to be related to some situations which are semantic correlates of certain past-tense (future-tense) sentences in the actual state of the world.

Let f be a function such that for any $t, t_1 \in T$, $\Gamma_t \in \mathcal{S}_t$: $f(\Gamma_t, t_1)$ is a non-empty subset of $\mathcal{S}_{t_1}$ such that:

(i) for any $\Gamma_{t_1}, \Delta_{t_1} \in f(\Gamma_t, t_1)$ if $\Gamma_{t_1} \leq \Delta_{t_1}$ or $\Delta_{t_1} \leq \Gamma_{t_1}$, then $\Gamma_{t_1} = \Delta_{t_1}$;

(ii) if $\Gamma_{t_1} \in f(\Gamma_t, t_1)$, then $\Gamma_t \in f(\Gamma_{t_1}, t)$;

(iii) for any Γ_t, Δ_t if $\Gamma_t \leq \Delta_t$, then $f(\Gamma_t, t_1)$ is non-empty and for any $\Gamma_{t_1} \in f(\Gamma_t, t_1)$ there is $\Delta_{t_1} \in f(\Delta_t, t_1)$ such that $\Gamma_{t_1} \leq \Delta_{t_1}$.

The condition (i) means that neither of two possible – from n via the set of situations Γ_t – sets of situations at t_1 can be bigger than the other. This condition simply means that sets of situations possible from n via Γ_t are not in the relation of set-theoretical inclusion. The condition (ii) assumes that if the set of situations Γ_{t_1} is possible at t_1 from n via Γ_t, then the set Γ_t is possible at t from n via Γ_{t_1}. The condition (iii) expresses the fact that if at t there are possible two sets of situations Γ_t and Δ_t and Γ_t is smaller than Δ_t, then at t_1 there is a set of situations

Γ_{t_1} possible from n via Γ_t and there is a set of situations Δ_{t_1} bigger than Γ_{t_1}, which is possible from n via Δ_t.

We assume that the function f is given for any world $\mathfrak{W}$. Hence formally $\mathfrak{W}$ is a 4-tuple $<W, R, f, n>$.

The sets of situations which form W_t are precisely those sets which are (now) possible at t. Thus – in consequence – precisely those facts take place at t which are determined at n.[6]

Since situations are semantic correlates of sentences, a propositional language will be enough for our purpose.

By propositional language L_P we understand the set of all formulas built up with the following symbols:

(i)	$p_0, p_1, \ldots$	– propositional letters
(ii)	$\neg$	– unary connective
(iii)	$\vee, \wedge, \rightarrow,$	– binary connectives
(iv)	), (	– parentheses (for punctuation)

By tense-logical language L we understand the set of all formulas with the above symbols and the following ones:

(v)	H, G, P, F	– unary connectives (tense operators).

As usual, H, G, P, F are read: "it has always been the case that", "it will always be the case that", "it has been the case that", "it will be the case that", respectively.

Usual conventions for omitting of parentheses are accepted. Instead of $p_0, p_1, \ldots$ we shall write $p, q, \ldots$ Formulas will be denoted by α, β, γ.

Now we have to answer what it means that a situation is determined by a set of situations. Let $\models$ denote the relation of determination.

For any $\alpha, \beta \in L$, $\Gamma \in \mathcal{S}_t$:

(i) if $\Gamma \Vdash \alpha$, and $\Gamma \leq \Delta$, then $\Delta \Vdash \alpha$,

(ii) $\Gamma \Vdash (\alpha \wedge \beta)$ iff $\Gamma \Vdash \alpha$ and $\Gamma \Vdash \beta$,

(iii) $\Gamma \Vdash (\alpha \vee \beta)$ iff $\Gamma \Vdash \alpha$ or $\Gamma \Vdash \beta$,

(iv) $\Gamma \Vdash (\neg\alpha)$ iff for all $\Delta \in \mathcal{S}_t$ such that $\Gamma \leq \Delta$, $\Delta \Vdash \alpha$ does not hold,

(v) $\Gamma \Vdash (\alpha \to \beta)$ iff for all $\Delta \in \mathcal{S}_t$ such that $\Gamma \leq \Delta$, if $\Delta \Vdash \alpha$, then $\Delta \Vdash \beta$,

(vi) $\Gamma \Vdash H\alpha$ iff for all t_1: if $t_1 Rt$, then for any $\Delta \in f(\Gamma, t_1)$, $\Delta \Vdash \alpha$;

(vii) $\Gamma \Vdash G\alpha$ iff for all t_1: if tRt$_1$, then for any $\Delta \in f(\Gamma, t_1)$, $\Delta \Vdash \alpha$;

(viii) $\Gamma \Vdash P\alpha$ iff there is t_1 such that $t_1 Rt$ and for some $\Delta \in f(\Gamma, t_1)$, $\Delta \Vdash \alpha$;

(ix) $\Gamma \Vdash F\alpha$ iff there is t_1 such that tRt_1 and for some $\Delta \in f(\Gamma, t_1)$, $\Delta \Vdash \alpha$.

The clauses (i)–(v) are the usual clauses for forcing.[7]

The relation $\Gamma \Vdash \alpha$ can be read: α is (now) determined relatively to $\Gamma (\in \mathcal{S}_t)$.

$W_t \models \alpha$ (α is valid in W_t, or α is determined in W_t) iff for any $\Gamma \in \mathcal{S}_t$:

$$\Gamma \Vdash \alpha$$

Let us remark that:

if there is t_1 such that $t_1 Rt$ and $W_{t_1} \models \alpha$, then $W_t \models P\alpha$

if there is t_1 such that tRt$_1$ and $W_{t_1} \models \alpha$, then $W_t \models F\alpha$

but in any case it is not true that:

if $W_t \models P\alpha$, then there is t_1 such that $t_1 Rt$ and $W_{t_1} \models \alpha$

if $W_t \models F\alpha$, then there is t_1 such that tRt$_1$ and $W_{t_1} \models \alpha$

E.g., it can be such that $W \models P(p \vee q)$ but for no $t_1, t_1 Rt$:

$$W_{t_1} \models p \vee q.$$

And similarly for F.

W_t will be called an I-model of α iff α is valid by the clauses (i)–(v). zW_t will be called an I-model of Σ iff for all $\alpha \in \Sigma$, W_t is an I-model of α. W_t is a counter-I-model of α iff W_t is not an I-model of α.

Let us say that Γ is an origin of W_t if $\Gamma \leq \Delta$, for any $\Delta \in \mathcal{S}_t$.

Note that:

(vi)' $W_t \models H\alpha$ iff for all t_1 such that t_1Rt, $W_{t_1} \models \alpha$

(vii)' $W_t \models G\alpha$ iff for all t_1 such that tRt_1, $W_{t_1} \models \alpha$

and if W_t has an origin, then:

(viii)' $W_t \models P\alpha$ iff there is t_1 such that t_1Rt, $W_{t_1} \models \alpha$

(ix)' $W_t \models F\alpha$ iff there is $t_1$such that tRt$_1$, $_{t_1} \models \alpha$.

Thus it is clear that our intuitionistic conceiving of H, G, P, F is related to the Priorian definition of these connectives.

It is easy to see that:

$W_t \models \neg P\alpha$ iff $W_t \models H\neg\alpha$;

$W_t \models \neg F\alpha$ iff $W_t \models G\neg\alpha$;

Now, we consider a tense logic which reflects the view that neither the thesis of predeterminism nor the thesis of postdeterminism [i.e., the thesis that one of the two sentences:

It has been the case that α;

It has been the case that not$-\alpha$,

where α does not refer to a future fact; will always be true (determined)] hold. The thesis of postdeterminism is symmetrical to the thesis of predeterminism. For this reason the function played by the designated element n is not formally essential.[8]

$<W, R> \models \alpha$ (α is valid in $<W, R>$) iff for any $t \in T$, $W_t \models \alpha$.

$<W, R>$ will be called a model of α iff $<W, R> \models \alpha$. α is satisfiable in $<W, R>$ iff there is $W_t \in W$ such that $W_t \models \alpha$. By a frame we understand $<T, R>$. A formula α is valid in a frame $<T, R>$ iff it is valid in every model $<W, R>$ based on this frame. Let $\mathcal{K}$ be a class of frames $<T, R>$. A formula α is valid over $\mathcal{K}$ iff it is valid in every frame $<T, R> \in \mathcal{K}$. A system $\mathfrak{S}$ is sound for $\mathcal{K}$ if every thesis of $\mathfrak{S}$ is valid over $\mathcal{K}$ and a sound system $\mathfrak{S}$ is complete for $\mathcal{K}$ if conversely every formula valid over $\mathcal{K}$ is a thesis of $\mathfrak{S}$.

α is a tautology of the minimal intuitionistic tense logic iff it is valid in every frame $<T, R>$; i.e., for any $<W, R>$:

$<W, R> \models \alpha$.

The minimal intuitionistic tense logic is the set of all formulas of L valid in every world $\mathfrak{W}$.

We shall write: $\Sigma \models \alpha$ (α is a semantic consequence of Σ) iff for all $<W, R>$, if $<W, R> \models \Sigma$, then $<W, R> \models \alpha$.

The minimal intuitionistic tense logic is axiomatizable.

2. MINIMAL INTUITIONISTIC TENSE LOGIC T_m

Axioms:

Besides an axiom system for the intuitionistic propositional logic (IPC), say

for all $\alpha, \beta, \gamma \in L$:

(1) $\alpha \to (\beta \to \alpha)$

(2) $(\alpha \to \beta) \to \{[\alpha \to (\beta \to \gamma)] \to (\alpha \to \gamma)\}$

(3) $[(\alpha \to \gamma) \wedge (\beta \to \gamma)] \to \{(\alpha \vee \beta) \to \gamma]$

(4) $(\alpha \wedge \beta) \to \alpha$

(5) $(\alpha \wedge \beta) \to \beta$

(6) $\alpha \to [\beta \to (\alpha \wedge \beta)]$

(7) $\alpha \to (\alpha \vee \beta)$

(8) $\beta \to (\alpha \vee \beta)$

(9) $(\alpha \wedge \neg\alpha) \to \beta$

(10) $(\alpha \to \neg\alpha) \to \neg\alpha$

and the rule of detachment (Modus Ponens):

MP. to infer β from α and $\alpha \to \beta$;

Tense-logical axioms:

(H1) $H(\alpha \to \beta) \to (H\alpha \to H\beta)$

(H1') $H(\alpha \to \beta) \to (P\alpha \to P\beta)$

(H1") $(H\alpha \to P\alpha) \vee H\beta$

(G1) $G(\alpha \to \beta) \to (G\alpha \to G\beta)$

(G1') $G(\alpha \to \beta) \to (F\alpha \to F\beta)$

(G1”)	$(G\alpha \rightarrow F\alpha) \vee G\beta$
(H2)	$\alpha \rightarrow HF\alpha$
(H2’)	$PG\alpha \rightarrow \alpha$
(G2)	$\alpha \rightarrow GP\alpha$
(G2’)	$FH\alpha \rightarrow \alpha$
(HD)	$P\alpha \rightarrow \neg H \neg \alpha$
(HD’)	$\neg P\alpha \rightarrow H\neg\alpha$
(GD)	$F\alpha \rightarrow \neg G \neg \alpha$
(GD’)	$\neg F\alpha \rightarrow G\neg\alpha$

Tense-logical rules:

(RH)	to infer $H\alpha$ from α
(RG)	to infer $G\alpha$ from α.

In the case of tense logic based on classical logic H1, G1, H2, G2 are the only axioms of minimal tense logic K_t. The operators H, P and G, F are dual. Thus P and F are definable: $P\alpha \equiv \neg H \neg \alpha; F\alpha \equiv \neg G \neg \alpha$. H1’, H1” and G1’, G1” are classical consequences of H1 and G1, respectively. It can be proved that the axioms H1–GD’ form an independent axiom system (on the basis of intuitionistic propositional calculus).

Theorem 1. T_m is consistent.

This fact immediately follows from the consistency of K_t, the minimal classical tense logic, since $T_m \subset T_k$.

The standard notion of proof will be assumed; i.e., a formula α of a language $\mathcal{L}$ has a proof from a set Σ of formulas of $\mathcal{L}$ based on a logic $\mathcal{C}$, $\Sigma \vdash_{\mathcal{C}} \alpha$, iff there exists a finite sequence of formulas of $\mathcal{L}$ such that the last element of the sequence is α and each element is either an axiom of $\mathcal{C}$ or an element of Σ, or is obtained from previous elements by use of the rules of inference of the logic $\mathcal{C}$.

Let Σ be a set of formulas of $\mathcal{L}$. Σ is a theory based on a logic $\mathcal{C}$ iff for any $\alpha \in \mathcal{L}$, $\alpha \in \Sigma$ iff $\Sigma \vdash_{\mathcal{C}} \alpha$.

If it is clear from a context we shall simply say: Σ is a theory, if

this theory is based on the minimal intuitionistic tense logic and we shall say that Σ is an intuitionistic theory, if this theory is based on the intuitionistic logic alone.

$Cn_{MP}\Sigma$ denotes the least set of formulas $\alpha \in L$ such that there is a MP-derivation $\alpha_0, \alpha_1, \ldots, \alpha_n (= \alpha)$ of α in the following sense:
for any i one of the conditions holds

(i) α_i is a member of Σ;

(ii) α_i is a thesis of T_m;

(iii) there are $j, k < i$ such that $\alpha_k = \alpha_j \to \alpha_i$.

Note that $Cn_{MP}\Sigma$ differs from $Cn\Sigma$, the set of formulas derivable from Σ by means of logic T_m, because in the definition of $Cn_{MP}\Sigma$ two rules of T_m, RH and RG, are not taken into consideration.

Let for a given set of formulas Σ, $\Sigma_G = \{\alpha : G\alpha \in \Sigma\}$; $\Sigma_G(\alpha) = \Sigma_G \cup \{\alpha\}$, if $F\alpha \in \Sigma$; $G\Sigma_G(F\alpha) = \{G\beta : \beta \in \Sigma_G(\alpha)$ and $\beta \neq \alpha\} \cup \{F\alpha\}$, $G\Sigma_G = \{G\alpha : \alpha \in \Sigma_G\}$.

Let for a given set of formulas Σ, $\Sigma_H = \{\alpha : H\alpha \in \Sigma\}$; $\Sigma_H(\alpha) = \Sigma_H \cup \{\alpha\}$, if $P\alpha \in \Sigma$; $H\Sigma_H(P\alpha) = \{H\beta : \beta \in \Sigma_H(\alpha)$ and $\beta \neq \alpha\} \cup \{P\alpha\}$, $H\Sigma_H = \{H\beta : \beta \in \Sigma_H\}$.

Lemma 1. For any $\beta \in Cn_{MP}\Sigma_G(\alpha), G\beta$ or $G(\alpha \to \beta)$ is a member of $Cn_{MP}G\Sigma_G(F\alpha)$.

Proof (by induction on the length of MP-derivation).

Let β be MP-derivable from $\Sigma_G(\alpha)$. Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ (= β) be a MP-derivation of β. If α_i is a member of $\Sigma_G(\alpha)$ or an axiom the thesis is clear. Let us assume that the thesis holds for $\alpha_j, j < i$. Let us consider all the possibilities of use of MP. Let $\alpha_k (= \alpha_l \to \alpha_i)$; $k, l < i$.

(i) $G\alpha_k, G\alpha_l \in Cn_{MP}G\Sigma_G(F\alpha)$. From $G1$ $G\alpha_l \to G\alpha_i \in Cn_{MP}G\Sigma_G(F\alpha)$ and hence by MP $G\alpha_i \in Cn_{MP}G\Sigma_G(F\alpha)$;

(ii) $G\alpha_k, G(\alpha \to \alpha_l) \in Cn_{MP}G\Sigma_G(F\alpha)$.

By twice applying MP to:

$$T_m \vdash G(\alpha_l \to \alpha_i) \to [G(\alpha \to \alpha_l) \to G(\alpha \to \alpha_i)]$$

we infer that

$$G(\alpha \to \alpha_i) \in Cn_{MP}G\Sigma_G(F\alpha);$$

(iii) $$G(\alpha \to \alpha_k), G\alpha_l \in Cn_{MP}G\Sigma_G(F\alpha).$$

By twice applying MP to:

$$T_m \vdash G[\alpha \to (\alpha_l \to \alpha_i)] \to G\alpha_l \to G(\alpha \to \alpha_i)]$$

we infer that

$$G(\alpha \to \alpha_i) \in Cn_{MP}G\Sigma_G(F\alpha);$$

(iv) $$G(\alpha \to \alpha_k), G(\alpha \to \alpha_l) \in Cn_{MP}G\Sigma_G(F\alpha).$$

By twice applying MP to:

$$T_m \vdash G[\alpha \to (\alpha_l \to \alpha_i)] \to [G(\alpha \to \alpha_l) \to G(\alpha \to \alpha_i)]$$

we infer that

$$G(\alpha \to \alpha_i) \in Cn_{MP}G\Sigma_G(F\alpha). \dashv$$

The proof of the next lemma is analogous.

Lemma 1'. For any $\beta \in Cn_{MP}\Sigma_H(\alpha)$, $H\beta$ or $H(\alpha \to \beta)$ is a member of $Cn_{MP}H\Sigma_H(P\alpha)$.

Lemma 1a. For any $\alpha \in Cn_{MP}\Sigma_G$, $G\alpha \in Cn_{MP}G\Sigma_G$.

Lemma 1a'. For any $\alpha \in Cn_{MP}\Sigma_H$, $H\alpha \in Cn_{MP}H\Sigma_H$.

The proofs of these lemmas are easy obtainable by the same reasons as in the point (i) in the proofs of Lemma 1 and Lemma 1', respectively.

Let us call a set of formulas Σ MP-consistent iff for no α, α and $\neg\alpha$ are MP-derivable from the set Σ.

Lemma 2. Let Σ be an MP-consistent set of formulas. If for some α, $F\alpha \in \Sigma$, then $\Sigma_G(\alpha)$ and Σ_G are MP-consistent.

Proof

Let $\Sigma_G(\alpha)$ be MP-inconsistent. Thus $\neg\alpha$ is MP-derivable from $\Sigma_G(\alpha)$. By Lemma 1 (i) $G(\alpha \to \neg\alpha)$ or (ii) $G\neg\alpha$ is a member of $Cn_{MP}G\Sigma_G(F\alpha)$. The case (i) is reducible to the case (ii). From

axiom (10) by RG and from $G1$ by MP it follows:

$$G(\alpha \to \neg\alpha) \to G\neg\alpha \in T_m.$$

Hence by MP $G\neg\alpha \in Cn_{MP}G\Sigma_G(F\alpha)$. Since $F\alpha \in Cn_{MP}G\Sigma_G(F\alpha)$, by GD

$$\neg G\neg\alpha \in Cn_{MP}G\Sigma_G(F\alpha).$$

Thus Σ is inconsistent because $\neg G\neg\alpha$ and $G\neg\alpha$ are members of $Cn_{MP}G\Sigma_G(F\alpha)$ and $Cn_{MP}G\Sigma_G(F\alpha) \subseteq Cn_{MP}\Sigma$. $\dashv$

If for some α, $F\alpha \in \Sigma$, then MP-inconsistency of Σ_G implies MP-inconsistency of Σ. It is out of place if for no α, $F\alpha \in \Sigma$.

The proof of the next lemma is analogous.

Lemma 2'. Let Σ be MP-consistent. If for some α, $P\alpha \in \Sigma$, then $\Sigma_H(\alpha)$ and Σ_H are MP-consistent.

Sets $Cn_{MP}\Sigma_G$ and $Cn_{MP}\Sigma_G(\alpha)$ will be counted as future attendants of $Cn_{MP}\Sigma$ iff there is some α such that $F\alpha \in Cn_{MP}\Sigma$. $Cn_{MP}\Sigma_H$ and $Cn_{MP}\Sigma_H(\alpha)$ will be counted as past attendants of $Cn_{MP}\Sigma$ iff there is some α such that $P\alpha \in Cn_{MP}\Sigma$.

Lemma 3. If Σ_1 is a future attendant of $\Sigma(= Cn_{MP}\Sigma)$, then:

(i) if $\alpha \in \Sigma_1$, then $F\alpha \in \Sigma$;

(ii) if $\alpha \in \Sigma$, then $P\alpha \in \Sigma_1$;

(iii) if $H\alpha \in \Sigma_1$, then $\alpha \in \Sigma$.

Proof

(i) By Lemma 1 (a) $G(\beta \to \alpha) \in \Sigma$ or (b) $G\alpha \in \Sigma$. In the case (a) from $G(\beta \to \alpha) \to (F\beta \to F\alpha)$, (G1'), by twice applying MP it follows that $F\alpha \in \Sigma$. In the case (b) because there is β such that $F\beta \in \Sigma$ and since $G\alpha \to (F\beta \to F\alpha) \in T_m$ we get $F\alpha \in \Sigma$.

(ii) If $\alpha \in \Sigma$, then by G2 $GP\alpha \in \Sigma$ and hence by definition of Σ_1, $P\alpha \in \Sigma_1$.

(iii) By (i) we have $FH\alpha \in \Sigma$ and hence by G2', $\alpha \in \Sigma$. $\dashv$

Lemma 3'. If Σ_1 is a past attendant of $\Sigma(= Cn_{MP}\Sigma)$, then:

(i) if $\alpha \in \Sigma_1$, then $P\alpha \in \Sigma$;

(ii) if $\alpha \in \Sigma$, then $F\alpha \in \Sigma_1$;

(iii) if $G\alpha \in \Sigma_1$, then $\alpha \in \Sigma$.

Take the set $W^*(\Sigma)$ to consist of $Cn_{MP}\Sigma$ and the past and future attendants of $Cn_{MP}\Sigma$, their attendants, the attendants of their attendants, etc. The members of $W^*(\Sigma)$ can be alphabetically ordered, so $Cn_{MP}\Sigma$ will now be referred to as $(Cn_{MP}\Sigma)_n$, and the other members of $W^*(\Sigma)$ are $\Sigma^*_{t_1}$, $\Sigma^*_{t_2}$, etc. This ordering of the members of $W^*(\Sigma)$ – and assigning them indices based upon it – is necessitated by the fact that two such sets may have identical memberships.

It can be proved that the set $W^*(\Sigma)$ is denumerable and thus a Löwenheim-Skolem-Theorem will be available for T_m.

Below an indexed set will be called a future (past) attendant of another if this set is a future (past) attendant of the other.

Lemma 4. (a) If Σ_1 is a future attendant of Σ, then Σ is a past attendant of Σ_1 and (b) if Σ is a past attendant of Σ_1, then Σ_1 is a future attendant of Σ.

Proof

(a) Let Σ_1 be a future attendant of Σ. By the definition there is α such that $F\alpha \in \Sigma$. By Lemma 3 (iii) if $H\beta \in \Sigma_1$, then $\beta \in \Sigma$. We have also to point out that there is γ such that $P\gamma \in \Sigma_1$. $GPF\alpha \in \Sigma$ since $F\alpha \in \Sigma$. Hence $PF\alpha \in \Sigma_1$. The proof of (b) is analogous. ⊣

Define R^*, a dyadic relation on the members of $W^*(\Sigma)$, as follows:

$$\Sigma^*_{t_1} R^* \Sigma^*_{t_2} \text{ if and only if } \Sigma^*_{t_2} \text{ is a future attendant of } \Sigma^*_{t_1}.$$

Lemma 5.

(i) If for every α, $H\alpha \in \Sigma$ and for some β, $P\beta \in \Sigma$, then Σ is MP-inconsistent.

(ii) If for every α, $G\alpha \in \Sigma$ and for some β, $F\beta \in \Sigma$, then Σ is MP-inconsistent.

Proof

(i) From the axiom HD and $P\beta$ by MP we get $\neg H\neg\beta$. There is a contradiction since by the assumption $H\neg\beta \in \Sigma$. The proof of (ii) is

analogous. ⊣

Lemma 6.

(i) If for every α, $H\alpha \in \Sigma$ and for some β, $F\beta \in \Sigma$, then Σ is inconsistent.

(ii) If for every α, $G\alpha \in \Sigma$ and for some β, $P\beta \in \Sigma$, then Σ is inconsistent.

Proof
If $F\beta \in \Sigma$, then for any $H\alpha \in \Sigma$ – with the help of the rule RG – $FH\alpha$ is derivable from Σ. Thus from $FH\alpha$ and G2' we get α and from $FH\neg\alpha$ and G2' we obtain $\neg\alpha$. The proof of (ii) is analogous. ⊣

Note that the inconsistency of Σ does not imply the MP-inconsistency of Σ.

Lemma 7. Let Σ be a consistent theory.

(i) If for some α, β: $F\alpha \in \Sigma, P\beta \in \Sigma$, then for any α, $\alpha \notin \Sigma$ there is $<W, R, W_n>$ such that: $W_n \models \Sigma$ and it is not true that $W_n \models \alpha$;

(ii) if for some α, $P\alpha \in \Sigma$ and for no α, $F\alpha$ is MP-derivable from $\Sigma \cup \{G\alpha : \alpha \in L\}$, then for any α, $\alpha \notin Cn_{MP}(\Sigma \cup \{G\alpha : \alpha \in L\}$ there is $<W, R, W_n>$ such that: $W_n \models \Sigma$ and it is not true that $W_n \models \alpha$;

(iii) if for some α, $F\alpha \in \Sigma$ and for no α, $P\alpha$ is MP-derivable from $\Sigma \cup \{H\alpha : \alpha \in L\}$, then for any α, $\alpha \notin Cn_{MP}(\Sigma \cup \{H\alpha : \alpha \in L\}$ there is $<W, R, W_n>$ such that: $W_n \models \Sigma$ and it is not true that $W_n \models \alpha$;

(iv) if for any α, neither $P\alpha$ nor $F\alpha$ is MP-derivable from $\Sigma \cup \{H\alpha : \alpha \in L\} \cup \{G\alpha : \alpha \in L\}$, then for any α, $\alpha \notin Cn_{MP}(\Sigma \cup \{H\alpha : \alpha \in L\} \cup \{G\alpha : \alpha \in L\}$ there is $<W, R, W_n>$ such that: $W_n \models \Sigma$ and it is not true that $W_n \models \alpha$; (or – what is equivalent – if $\alpha \notin Cn(\Sigma \cup \{H\alpha : \alpha \in L\} \cup \{G\alpha : \alpha \in L\})$, then it is not true that $W_n \models \alpha$).

Proof

(i) Because: $T \vdash P\alpha \rightarrow (H\beta \rightarrow P\beta), T_m \vdash F\alpha \rightarrow (G\beta \rightarrow F\beta)$ we have:

$$H^{-\infty}\, H\alpha \rightarrow P\alpha \in \Sigma$$

$$G^{+\infty}\, G\alpha \rightarrow F\alpha \in \Sigma.$$

Every formula of the types: $\sigma(H\beta \rightarrow P\beta)$, $\sigma(G\beta \rightarrow \beta)$ where σ is a finite sequence of H and G, is a member of Σ. Thus for any $\Sigma^*_{t_1} \in W^*(\Sigma)$, $H^{-\infty}$ and $G^{+\infty}$ are members of $\Sigma^*_{t_1}$. Hence it is clear that for any $\Sigma^*_{t_1}$ there is $\Sigma^*_{t_2}$ such that $\Sigma^*_{t_1}R^*\Sigma^*_{t_2}$ and there is $\Sigma^*_{t_3}$ such that $\Sigma^*_{t_3}R^*\Sigma^*_{t_1}$. In the case (ii) Σ does not have any future attendant, in the case (iii) Σ does not have any past attendant and in the case (iv) Σ have neither past nor future attendants. In each case (i)–(iv) the triple $<W, R, W_n>$ such that:

it is not true that $W_n \models \alpha$, if $\alpha \notin \Sigma$

is constructed inductively. It is known that intuitionistic calculus is strongly complete with respect to the class of Kripke models with origins. Hence if $\alpha \notin \Sigma$, then there is an I-model (of Σ) with an origin Γ such that $\Gamma \models \alpha$ does not hold. Let W_n be such an I-model of Σ. For any $\Gamma \in \mathcal{S}_n$ it have to be such that $\Gamma \models \alpha$ iff α is an MP-consequence of Σ and the set of all formulas β such that $\Gamma \models \beta$ (in the I-model of Σ). Let $\Sigma^*_{t_2}$ be a future attendant of $\Sigma^*_{t_1}$. Let us take all the future attendants of the set of formulas determined at Γ_{t_1} which contain $\Sigma^*_{t_2}$, for any $\Gamma_{t_1} \in \mathcal{S}_{t_1}$. Let $f(\Gamma_{t_1}, t_2)$ be the set of all these future attendants. It can be checked that we obtain an I-model of $\Sigma^*_{t_2}$ if $\mathcal{S}_{t_2}$ is the set of all elements of $f(\Gamma_{t_1}, t_2)$, for any $\Gamma_{t_1} \in \mathcal{S}_{t_1}$, and the relation $\leq$ is defined by the set-theoretical inclusion between sets of formulas. The construction of past attendants is analogous. From the construction we have that any W_t is an I-model of Σ^*_t and also that for any Γ_t:

(i) $\Gamma_t \Vdash F\alpha(P\alpha)$ iff there is W_{t_1}, a future (past) attendant of W_t, such that for some $\Gamma_{t_1} \in f(\Gamma_t, t_1)$, $\Gamma_{t_1} \Vdash \alpha$;

(ii) $\Gamma_t \Vdash G\alpha(H\alpha)$ iff for any W_{t_1}, a future (past) attendant of W_t, if $\Gamma_{t_1} \in f(\Gamma_t, t_1)$, then $\Gamma_{t_1} \Vdash \alpha$. $\dashv$

Lemma 8. $\Sigma \vdash \alpha$ iff α is derivable from $\Sigma_1 (= \Sigma \cup H^{-\infty} \cup G^{+\infty})$ and α is MP-derivable from $\Sigma_2 (= Cn(\Sigma \cup H^{-\infty}) \cup \{G\alpha : \alpha \in L\})$, $\Sigma_3 (= Cn(\Sigma \cup G^{+\infty}) \cup \{H\alpha : \alpha \in L\})$, $\Sigma_4 (= Cn\Sigma \cup \{H\alpha : \alpha \in$

$L\} \cup \{G\alpha : \alpha \in L\})$.

Proof

Of course, if $\Sigma \vdash \alpha$, then for any i, $1 \leq i \leq 4$, $\Sigma_i \vdash \alpha$.

Assume that α is derivable from Σ_1 and MP-derivable from Σ_i, $2 \leq i \leq 4$. In the MP-derivation of α from Σ_3 and from Σ_4, and in the MP-derivation of α from Σ_2 and from Σ_4 besides theses of Σ, i.e. formulas derivable from Σ, can be used formulas of the types $H\beta$ and $G\beta$, respectively. Because

$$T_m \vdash H(\alpha \wedge \beta) \equiv (H\alpha \wedge H\beta)$$

$$T_m \vdash G(\alpha \wedge \beta) \equiv (G\alpha \wedge G\beta)$$

we can point out γ and δ such that all the formulas of the types $H\beta$ and $G\beta$ used in the MP-derivation of α are MP-derivable from $H\gamma$ and $G\delta$, respectively. Of course, these formulas are also members of Σ_3, Σ_4 and Σ_2, Σ_4; respectively.

Let P^* and F^* be conjunctions of all the formulas of the types $H\beta \to P\beta$ and $G\beta \to F\beta$ used in the derivations of α from Σ_1, Σ_2 and from Σ_1, Σ_3, respectively.

It is easy to see that by the compactness theorem:

$$\Sigma \vdash H\gamma \wedge G\delta \to \alpha;$$

$$\Sigma \vdash G\delta \wedge P^* \to \alpha;$$

$$\Sigma \vdash H\gamma \wedge F^* \to \alpha;$$

$$\Sigma \vdash P^* \wedge F^* \to \alpha.$$

Hence

$$\Sigma \vdash [H\gamma \wedge G\delta \vee G\delta \wedge P^* \vee H\gamma \wedge F^* \vee P^* \wedge F^*] \to \alpha.$$

Because

$$\begin{aligned} T_m \quad \vdash \quad & [(H\gamma \vee P^*) \wedge (G\delta \vee F^*)] \\ \to \quad & [H\gamma \wedge G\delta \vee G\delta \wedge P^* \vee H\gamma \wedge F^* \vee P^* \wedge \ldots \wedge F^*] \end{aligned}$$

we get $\Sigma \vdash [(H\gamma \vee P^*) \wedge (G\delta \vee F^*)] \to \alpha$. Now we prove that: $T_m \vdash H\gamma \vee P^*$; $T_m \vdash G\delta \vee F^*$. Because

$$[(\alpha \vee \beta_1) \wedge \ldots \wedge (\alpha \vee \beta_n)] \to [\alpha \vee (\beta_1 \wedge \ldots \wedge \beta_n)]$$

is an intuitionistic thesis, thus if

$$P^* = (H\phi_1 \to P\phi_1) \wedge \ldots \wedge (H\phi_n \to P\phi_n),$$

$$F^* = (G\eta_1 \to F\eta_1) \wedge \ldots \wedge (G\eta_m \to F\eta_m),$$

then

$$\begin{aligned} T_m \ &\vdash \ [H\gamma \vee (H\phi_1 \to P\phi_1)] \wedge \ldots \\ &\wedge \ [H\gamma \vee (H\phi_n \to P\phi_n)] \to (H\gamma \vee P^*) \\ T_m \ &\vdash \ [G\delta \vee (G\eta_1 \to F\eta_1)] \wedge \ldots \\ &\wedge \ [G\delta \vee (G\eta_m \to F\eta_m)] \to (G\delta \vee F^*). \end{aligned}$$

Since all the formulas: $H\gamma \vee (H\phi_i \to P\phi_i), G\delta \vee (G\eta_i \to F\eta_i)$ are axioms of T_m we have $\Sigma \vdash \alpha$. $\dashv$

Theorem 2. (Completeness) $\Sigma \vdash \alpha$ iff $\Sigma \models \alpha$

Proof

By induction on the length of derivation we can prove that:

if $\Sigma \vdash \alpha$ then $\Sigma \models \alpha$.

Let us suppose that $\Sigma \vdash \alpha$ does not hold. Thus by Lemma 8 there is i such that for $i = 1, \alpha$ is not derivable and for $i > 1, \alpha$ is not MP-derivable from Σ_i. Σ_i fulfills one of the conditions of Lemma 7. If for some α, β: $\Sigma \vdash P\alpha$, $F\alpha$, then i = 1. If either $F\beta$ or $P\alpha$ or both does not follow from Σ, then by Lemma 5 and MP-consistency of Σ_i either i = 2 or i = 3 or i = 4, respectively. Hence by Lemma 7 there is $<W, R, W_n>$ such that $W_n \models \Sigma$ but $W_n \models \alpha$ does not hold. $\dashv$

Theorem 3. (FMP). α is a tautology of T_m iff α is valid in every $<W, R>$, such that:

(i) W has finite many elements,

(ii) for every $W_t \in W$, $\mathcal{S}_t$ has finite many elements.

Proof

Validity of α in every finite model immediately follows from the assumption that α is a tautology of the minimal intuitionistic tense logic. Thus we have to show that if α is not a tautology of T_m, then there is $<W, R, n>$ such that W and for every $W_t \in W$, $\mathcal{S}_t$ have finite many elements, and $<W, R, n> \models \alpha$ does not hold. By the assumption

there is W_n^1 such that $W_n^1 \models \alpha$ does not hold. At first we choose a finite class of members of W^1. Let W' be the least subset of W^1 such that

(i) $W_n^1 \in W'$,

(ii) if $W_t^1 \in W'$ and for some Γ_t, $\Gamma_t \Vdash P\beta$, $(\Gamma_t \Vdash F\beta)$, where $P\beta$ ($F\beta$, respectively) is a proper subformula of α, and for $\Delta \in \mathcal{S}_{t_1}^1$, $\Delta = f(\Gamma_t, t_1)$, $\Delta \Vdash \beta$, then $W_{t_1}^1 \in W'$,

(iii) if $W_t^1 \in W'$ and for some Γ_t, $\Gamma_t \Vdash H\beta$ $(G\beta)$ does not hold, where $H\beta$ ($G\beta$, respectively) is a proper subformula of α, and for $\Delta \in \mathcal{S}_{t_1}^1$, $\Delta = f(\Gamma_t, t_1)$, $\Delta \Vdash \beta$ does not hold, then $W_{t_1}^1 \in W'$.

By the construction of W', since the number of proper subformulas of a formula is finite, it is clear that W' has finitely many elements. It is also clear that W_n' has finitely many elements, and that $W_n' \models \alpha$ does not hold. $\dashv$

From this theorem it follows that:

Corollary (Decidability of T_m). The question whether α is a tautology of T_m is decidable.

3. MINIMAL POSTDETERMINISTIC TENSE LOGIC T_m^I

Above the tense logic reflecting the view that neither the thesis of post- nor the thesis of pre-determinism hold have been considered. Few philosophers maintain the view, e.g. J. Łukasiewicz. Most indeterminists reject only the thesis of predeterminism. They maintain that *quod fuit non potest non fuisse* – "what has been, cannot now not have been". Usually the thesis that the present state of the world is fully determined is also accepted. Such a view can be reflected in our semantics by the assumption that for any $t \leq n$ and any $\alpha \in L_P$ (i.e. α being a formula in which the tense operators do not occur), α or $\neg\alpha$ is determined. Now the world is asymmetric and n (now) fulfills an important function: n divides the completely determined part of the world from the not completly determined one. Thus we should modify the notion of validity.

α is valid in $<T, R, n>$ iff for every $<W, R, W_n>$ based on $<T, R, n>$, $W_n \models \alpha$.

α is a tautology of the minimal predeterministic tense logic T_m^I iff α is valid in every $<T, R, n>$.

Let T_m' be the axiom system consisting of all the axioms of T_m, the rule of detachment (MP) and (instead of RH and RG) the following axioms:

(AH) $H\alpha$, if α is an axiom

(AG) $G\alpha$, if α is an axiom.

Of course,

Lemma 9. $T_m \vdash \alpha$ iff $T_m' \vdash \alpha$.

Let T_m^I be the axiom system obtained by adding to T_m' the following axioms:

(A11) $\alpha \vee \neg\alpha$, if $\alpha \in L_P$

(AH') $H\alpha$, if α is an axiom.

Note that A11 together with A1–A10 and MP form an axiom system of the classical propositional calculus (CPC).

It is easy to prove that all the theses of T_m^I are tautologies of T_m^I.

Lemma 10. If $T_m^I \vdash \alpha$, then $T_m^I \models \alpha$.

Proof

All the theses of T_m', or what is the same by Lemma 9, all the theses of T_m are tautologies of T_m^I. This fact immediately follows from the definition of tautology of the minimal tense logic and soundness of T_m. All the axioms A11 and AH' are also tautologies. $\dashv$

The question whether all the tautologies of T_m^I are the theses of T_m^I remains unanswered.

Observe that $P\alpha \vee \neg P\alpha$ is a not tautology of T_m^I for all α. The law of excluded middle holds only if α does not refer to a future fact. E.g, $PG\alpha \vee \neg PG\alpha$ is not a tautology. But if the future tense operators G and F do not occur in α, then $P\alpha \vee \neg P\alpha$ is a tautology.

From the philosophical point of view the above considerations showed that on the basis of intuitionistic logic it is possible to construct a logic which reflects the view of indeterminists and moreover, in intuitively conceivable way, the philosophical convictions of indeterminists justify the content of some methods of intuitionistic logic, e.g. forcing.

4. INTUITIONISTIC TENSE LOGIC, FIRST-ORDER INTUITIONISTIC CALCULUS AND K_t

Let us now consider formal connections between the intuitionistic tense logic and intuitionistic first-order calculus. Thus we are interested in a translation taking tense-logical formulas to formulas of intuitionistic first-order calculus. A translation ought to be such that the tense-logical validity of a formula α is equivalent to the intuitionistic validity of the first-order formula being the translation of the formula α. Our semantic definitions are formulated in a non-formal language. This language, as usual, is conceived classically. Hence there are possible different classically equivalent definitions which do not have to be intuitionistically equivalent. The clause (vi) can be formulated equivalently as:

(vi') $\Gamma \Vdash H\alpha$ iff for any t_1: either $t_1 R t$ does not hold or for any $\Delta \in f(\Gamma_t, t_1)$, $\Delta \Vdash \alpha$

(vi) and (vi') are not intuitionistically equivalent. Nevertheless the given semantic definition can be viewed as pointing out by the right translation. Tense-logical formulas are translated into formulas of first-order calculus having a binary predicate constant R as well as unary predicate constants P (one for each propositional letter p). The translation $\mathfrak{J}$ has a fixed free individual variable n (now).

(i) $\mathfrak{J}(p) = P(n)$

(ii) $\mathfrak{J}(\neg\alpha) = \neg\mathfrak{J}(\alpha)$

(iii) $\mathfrak{J}(\alpha \rightarrow \beta) = \mathfrak{J}(\alpha) \rightarrow \mathfrak{J}(\beta)$

(iv) $\mathfrak{J}(\alpha \vee \beta) = \mathfrak{J}(\alpha) \vee \mathfrak{J}(\beta)$

(v) $\mathfrak{J}(\alpha \wedge \beta) = \mathfrak{J}(\alpha) \wedge \mathfrak{J}(\beta)$

(vi) $\mathfrak{J}(H\alpha) = \forall t\{\text{tRn} \rightarrow [t/n]\mathfrak{J}(\alpha)\}$

(vii) $\mathfrak{J}(P\alpha) = \exists t\{\text{tRn} \wedge [t/n]\mathfrak{J}(\alpha)\}$

(viii) $\mathfrak{J}(G\alpha) = \forall t\{\text{nRt} \rightarrow [t/n]\mathfrak{J}(\alpha)\}$

(ix) $\mathfrak{J}(F\alpha) = \exists t\{\text{nRt} \wedge [t/n]\mathfrak{J}(\alpha)\}$,

where $[t/n]\mathfrak{J}(\alpha)$ is the result of substituting some new temporal variable t (not occurring previously) for n in $\mathfrak{J}(\alpha)$.

T is a non-empty set of parameters. $\mathcal{P}$ is a map from $\mathcal{G}$ to non-empty sets of parameters. $\hat{\mathcal{P}}(\Gamma)$ denotes the set of all formulas which may be constructed using only parameters of $\mathcal{P}(\Gamma)$. $\Gamma^{\leq} \in \{\Delta : \Gamma \leq \Delta\}$. By a (first-order intuitionistic) model we mean $<\mathcal{G}, \leq, \models_I, \mathcal{P}>$, where $\mathcal{G}$ is a non-empty set, $\leq$ is a transitive, reflexive relation on $\mathcal{G}$, $\models_I$ is a relation between elements of $\mathcal{G}$ and formulas satisfying the following conditions:

For any $\Gamma \in \mathcal{G}$

(Q0) $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Gamma^{\leq})$,

(Q1) if $\Gamma \models \alpha$, then $\alpha \in \hat{\mathcal{P}}(\Gamma)$ for α atomic,

(Q2) if $\Gamma \models_{I'} \alpha$ then $\Gamma^{\leq} \models_I \alpha$ for α atomic,

(Q3) $\Gamma \models (\alpha \wedge \beta)$ iff $\Gamma \models_I \alpha$ and $\Gamma \models_I \beta$,

(Q4) $\Gamma \models_I (\alpha \vee \beta)$ iff $(\alpha \vee \beta) \in \hat{\mathcal{P}}(\Gamma)$ and $\Gamma \models_I \alpha$ or $\Gamma \models_I \beta$,

(Q5) $\Gamma \models_I \neg\alpha$ iff $\neg\alpha \in \hat{\mathcal{P}}(\Gamma)$ and for any $\Gamma^{\leq}$, $\Gamma^{\leq} \models \alpha$ does not hold,

(Q6) $\Gamma \models_I (\alpha \rightarrow \beta)$ iff $(\alpha \rightarrow \beta) \in \hat{\mathcal{P}}(\Gamma)$ and for all $\Gamma^{\leq}$, if $\Gamma^{\leq} \models_I \alpha$, then $\Gamma^{\leq} \models_I \beta$;

(Q7) $\Gamma \models_I (\exists x)\alpha(x)$ iff for some $t \in \mathcal{P}(\Gamma)$, $\Gamma \models_I \alpha(t)$,

(Q8) $\Gamma \models_I (\forall x)\alpha(x)$ iff for every $\Gamma^{\leq}$ and for every $t \in \mathcal{P}(\Gamma^{\leq})$, $\Gamma^{\leq} \models_I \alpha(t)$.

We call a particular formula α valid in the model

$$<\mathcal{G}, \leq, \models_I, \mathcal{P}>$$

if for all $\Gamma \in \mathcal{G}$ such that $\alpha \in \mathcal{P}(\Gamma)$, $\Gamma \models_I \alpha$. α is called valid if α is valid in all models.[9]

Lemma 11. For any $W_t \in W$ and $\Gamma \in \mathcal{S}_t$:

$$\Gamma \Vdash \alpha \text{ iff } \Gamma^{\leq} \Vdash \alpha,$$

where $\Gamma^{\leq} \in \{\Delta : \Gamma \leq \Delta\}$.

Proof A straightforward induction on the degree of formula.[10]

Let $f(\Gamma_t)$ be the least set such that:

(i) $\Gamma_t \in f(\Gamma_t)$

(ii) if $\Gamma_{t_1} \in f(\Gamma_t)$, then exactly one of the elements of $f(\Gamma_{t_1}, t_2)$ belongs to $f(\Gamma_t)$, for any $t_1, t_2 \in T$.

Let $f\mathfrak{W}$ be the least class of $f(\Gamma_t)$, for any $\Gamma_t \in \mathcal{S}_t$, $t \in T$.

Theorem 4. Let $\mathfrak{M}$ (= $<\mathcal{G}, \leq, \models_I, \mathcal{P}>$) be such that:

(i) $\mathcal{G} = f(\mathfrak{W}$;

(ii) $\Gamma \models_I P(t)$ iff there is $\Gamma_t \in \Gamma$ and $\Gamma_t \Vdash p$;

(iii) $\Gamma \leq \Delta$ iff $\Gamma_t \leq \Delta_t$, for some $\Gamma_t \in \Gamma$, $\Delta_t \in \Delta$;

(iv) $\Gamma \models_I tRt_1$ iff tRt_1.

$$<W, R, W_n> \models \alpha \text{ iff } \mathfrak{M} \models \mathfrak{I}(\alpha)$$

or, rather more exactly:

$$<W, R, W_n, f> \models \alpha \text{ iff } \mathfrak{M} \models \mathfrak{I}(\alpha).$$

Proof

From Lemma 11 and the definition of $\mathcal{G}$ it follows that $\mathfrak{M}$ is a first-order intuitionistic model. The relation between elements of $\mathcal{G}$ and atomic formulas is uniquely extended to the relation between elements of $\mathcal{G}$ and formulas. It can be shown that:

$$W_t \models \alpha \text{ iff } \mathfrak{M}[t/n]\mathfrak{I}(\alpha).$$

By the definition of $\mathfrak{M}$ we get:

$$W_t \models p \text{ iff } \mathfrak{M} \models [t/n]\mathfrak{I}(p)$$

Let us consider only the case of connective G.

If $W_t \models G\alpha$, then for any t_1: if tRt$_1$, then $W_{t_1} \models \alpha$. By the assumption for any t_1, $W_{t_1} \models \alpha$ iff $\mathfrak{M} \models [t_1/n]\mathfrak{J}(\alpha)$. Hence for any t_1, $\mathfrak{M} \models \{tRt_1 \rightarrow [t_1/n]\mathfrak{J}(\alpha)\}$. Thus $\mathfrak{M} \models \forall t_1\{tRt_1 \rightarrow [t_1/n]\mathfrak{J}(\alpha)\}$ and finally $\mathfrak{M} \models [t/n]\mathfrak{J}(G\alpha)$.

In order to show that if $\mathfrak{M} \models [n/t]\mathfrak{J}(G\alpha)$; then:

$$\text{for any } t_1, \text{ if } tRt_1, \text{ then } W_t \models G\alpha$$

we take the above proof in reverse order. ⊣

The translation motivates intuitionistic predicate calculus. E.g., $\neg\forall x\neg P(x) \rightarrow \exists x P(x)$ is "equivalent" to:

$$\neg G\neg p \rightarrow Fp.$$

$\neg G\neg\alpha \rightarrow F\alpha$ is a tautology iff the thesis of predeterminism holds, i.e. iff for any t, nRt:

$$W_t \models \alpha \vee \neg\alpha, \text{ if } \alpha \in L_P.$$

Let us now establish the connection between the minimal intuitionistic tense logic T_m and the minimal classical tense logic K_t.

Theorem 5. If H and G do not occur in α, then

$$T_m \vdash \alpha \text{ iff } K_t \vdash \neg\neg\alpha.$$

Proof

Of course, if $T_m \vdash \alpha$, then $K_t \vdash \neg\neg\alpha$,
since $T_m \subset K_t$.

Every theorem of K_t in which H and G do not occur is a thesis of K_t^*. K_t^* consists of – besides an axiom system for classical propositional calculus with MP as its sole rule of inference – the following axioms:

(H1*) $\neg P\neg(\alpha \rightarrow \beta) \rightarrow (\neg P\neg\alpha \rightarrow \neg P\neg\beta)$

(G1*) $\neg F\neg(\alpha \rightarrow \beta) \rightarrow (\neg F\neg\alpha \rightarrow \neg F\neg\beta)$

(H2*) $\alpha \rightarrow \neg P\neg F\alpha$

(G2*) $\alpha \rightarrow \neg F\neg P\alpha$

rules MP and:

(RH*) to infer $\neg P\neg\alpha$ from α

(RG*) to infer $\neg F \neg \alpha$ from α.

$H\alpha \equiv \neg P \neg \alpha$ and $G\alpha \equiv \neg F \neg \alpha$ are theorems of K_t. Thus K_t and $K_t^* \cup \{H\alpha \equiv \neg P \neg \alpha, G\alpha \equiv \neg F \neg \alpha\}$ are equivalent.

H1* is provable in T_m. From $\neg\neg(\alpha \rightarrow \beta) \rightarrow (\neg\neg\alpha \rightarrow \neg\neg\beta)$ by RH and from H1 we infer that:

$$H\neg\neg(\alpha \rightarrow \beta) \rightarrow (H\neg\neg\alpha \rightarrow H\neg\neg\beta)$$

and then, because:

$$\neg P \neg \alpha \equiv H \neg\neg\alpha,$$

we get

$$\neg P \neg(\alpha \rightarrow \beta) \rightarrow (\neg P \neg \alpha \rightarrow \neg P \neg \beta).$$

The proof that G1* is a thesis of T_m is similar. H2* and G2* are immediately obtainable from H2 and G2 since $HF\alpha \rightarrow \neg P \neg F\alpha$ and $GP\alpha \rightarrow \neg F \neg P\alpha$ are theses of T_m. Because H1*–G2* are theses of T_m their double negations are also theses of T_m. The double negation of the axioms of classical propositional calculus are theses of intuitionistic propositional calculus.

Let us now assume that α and $\alpha \rightarrow \beta$ are theses of K_t^* and $\neg\neg(\alpha \rightarrow \beta)$, $\neg\neg\alpha$ are theses of T_m:

(i) $\neg\neg\beta$ is a thesis of T_m, since $\neg\neg(\alpha \rightarrow \beta) \rightarrow (\neg\neg\alpha \rightarrow \neg\neg\beta)$ is a thesis of T_m;

(ii) $\neg\neg\neg P \neg \alpha$ is a thesis of T_m, since $H\neg\neg\alpha$ is a thesis of T_m and because $\neg P \neg \alpha \equiv H \neg\neg\alpha$;

by a similar argument

(iii) $\neg\neg\neg F \neg \alpha$ is a thesis of T_m. ⊣

The condition that H and G do not occur in α is essential, e.g.

$$\neg\neg G(\alpha \vee \neg\alpha)$$

is not a thesis of T_m, but

$$\neg P \neg(\alpha \vee \neg\alpha)$$

and

$$\neg F \neg(\alpha \vee \neg\alpha)$$

are theses of T_m no matter whether H and G occur or do not occur in α. Aristotle insists that $\alpha \vee \neg\alpha$ is always true, even if α refers to future

contingencies, although in this case neither α nor $\neg\alpha$ are true. Both the theses:

$$\neg P\neg(\alpha \vee \neg\alpha); \neg F\neg(\alpha \vee \neg\alpha)$$

say, roughly speaking, that $\alpha \vee \neg\alpha$ is never false. $\neg P\neg\alpha \to H\alpha$ and $\neg F\neg\alpha \to G\alpha$ can be viewed as formulations of the theses of post- and pre-determinism, respectively.

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NOTES

[1] In "*On Determinism*" Łukasiewicz discusses two arguments in favour of the thesis of determinism. He writes: "However, there are two arguments of considerable persuasive power which have been known for a long time and which seem to support determinism. One of them, originating with Aristotle, is based on the logical principle of the excluded middle, and the other, which was known to Stoics, on the physical principle of causality." (p. 114) ... "Although the argument based on the principle of the excluded middle is independent of that derived from the principle of causality, the former indeed becomes fully intelligible if every fact has its causes existing from all eternity." (p. 122).

[2] At the beginning of the paper '*Consciousness, philosophy and mathematics*' Brouwer writes: "In many respects it (intuitionistic mathematics) deviates from classical mathematics. In the first place because classical mathematics uses logic to generate theorems, believes in the existence of unknown truths, and in particular applies *the principle of the excluded third* expressing that every mathematical assertion (i.e. every assignment of a mathematical property to a mathematical entity) either is a truth or cannot be a truth. In the second place because classical mathematics confines itself to *predeterminate* infinite sequences for which from the beginning the nth element is fixed for each n. Owing to this confinement classical mathematics, to define real numbers, has only predeterminate convergent infinite sequences of rational numbers at its disposal."(p. 78)

[3] L. E. J. Brouwer [1964a] writes: "This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely;

this gives rise still further to the smallest infinite ordinal number ω. Finally this basal intuition of mathematics, in which the connected and separate, the continuous and discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the 'between', which is not exhaustible by the interpretation of new units and which therefore can never be thought of as a mere collection of units." (p. 69)

[4] It is well known that some notions of intuitionism are intuitively motivated via tense logic, e.g. Smorynski C., *Investigations of Intuitionistic* ..., p. 20; Grzegorczyk [1964].

[5] Łukasiewicz's many-valued approach has some counter-intuitive consequences, e.g. under the

three-valued matrix $M(\alpha \wedge \neg\alpha)$ gets the value 1. This fact was already observed by Gonseth on the conference in Zurich in 1938. A. N. Prior [1967] writes: "But that logic (i.e., Łukasiewicz's 3-valued logic) has some features which are very counter-intuitive even when we do take the possibility of 'neuter' propositions seriously; in particular, a conjunction of two neuter propositions is neuter, even in the case where one is the negation of the other. If 'There will be a sea-battle' is neuter or undecided, it is no doubt reasonable that 'There will be no sea-battle' should be neuter or undecided too; but not that 'There both will and won't be a sea-battle' should be – that, surely, is plain false. On the other hand, it is equally unplausible to make the conjunction of two neuters automatically false; if they're independent, it is natural that their conjunction should be neuter too. The truth-functional technique seems simply out of place here." (p. 135). Łukasiewicz was not moved by this counter-example. A few years before his death he writes that he cannot find any example that refutes his concepts of necessity and possibility. "On the contrary, all seem to support its correctness", see, Łukasiewicz [1953]. Cf. also: R. Bull and K. Segerberg [1984], pp. 8–9. For a review of the problem of logical determinism and Łukasiewicz's three-valued logic see, e.g. Karpenko, A. S. [1984]

[6] Our semantics has not only a different motivation than e.g. the semantics for intuitionistic tense logic of Ewald [1986]. Important for the logic is the fact that in our case there are states of the world ordered by the relation of temporal precedence (no conditions are imposed on this relation). Members of the states of the world are sets ordered by the relation of set-theoretical inclusion – a refelexive and transitive relation (partial-order) but in the case of logic by Ewald "we have a partially-ordered set of 'states-of-knowledge', which we think of as belonging to a tense-logician who is studying a set of times. Within each state-of-knowledge there is a set of times and a temporal ordering." (p. 166)

[7] Our motivation for this semantics is near to the motivation given, e.g. in Fitting, M. C. [1969]

[8] J. Łukasiewicz [1970] writes: "We should not treat the past differently from the future. If the only part of the future that is now real is that which is causally determined by the present instant, and if causal chains commencing in the future belong to the realm of possibility, then only those parts of the past are at present real which still continue to act by their effects today. Facts whose effects have disappeared altogether and which even an omniscient mind could not infer from those now occurring, belong to the realm of possibility. One cannot say about them that they took place, but only that they were *possible*. It is well that it should be so. There are hard moments of suffering and still harder ones of guilt in everyone's life. We should be glad to be able to erase them not only from our memory but also from existence. We may believe that when all the effects of those fateful moments are exhausted, even should that happen only *after* our death, then their causes too will be effaced from the world of actuality and pass into the realm of possibility. Time calms our cares and brings us forgiveness." (pp. 127–128)

[9] Cf. Fitting [1969], p. 46.

[10] Cf. Fitting [1969], p. 47.

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PAWEŁ TURNAU

ASYMMETRICAL RELATIONS

I discuss and develop an argument set out by Russell in his *Theory of Knowledge* (1913). As far as I am aware it has not been given the consideration it deserves, even though it is directly relevant to recent discussions in semantics (involving theories based on the Fregean notion of sense (*Sinn*), and situation semantics. Apart from creating a difficulty for those theories, the argument may serve to undermine some popular views about relations. The success of the set-theoretical notion of relations, viz. as sets of ordered pairs, triples, etc., seems to have created the impression that relations can no longer be the subject of fruitful philosophical investigations. Russell's argument, however, shows that this notion is defective from the philosophical point of view.

Let us consider the following picture of reality. The world consists of objects, properties and relations. (We may as well dispense with properties if we treat them as monadic relations.) Any relation taken together with a suitable number of objects (depending on how many arguments the relation has) may form a complex, which may be real, and is then called a fact, or merely possible. In this picture, a relation is something which "cements" separate objects into one complex. A complex may also enter into other complexes, because a complex is a kind of object too, namely, a non-simple one. (We need not commit ourselves to the existence of simple objects: perhaps every object can be analyzed as consisting of a relation and other, simpler objects. This, however, need not concern us here.)

This picture of reality is strongly reminiscent of Russell's early theory of truth, the so-called "multiple relation theory of truth". It would be wrong, however, to think that its significance is purely historical. Thus, for instance, the notion of a relational complex on which it is based has been recently used by Barwise, Perry, Etchemendy, et al. More importantly, however, the argument that I am going to present does not depend upon this notion in any crucial way. It uses only some very basic properties of the notion of a relational complex, which it shares with other, more popular notions, such as that of a truth condition, or

Jan Woleński (ed.), Philosophical Logic in Poland, 297–305.

the sense of a sentence (Fregean *Gedanke*). The important property shared by these notions is that a relational complex, a truth condition, or a thought are all supposed to be wholes consisting of parts. Let me explain this point in Fregean terms. A contrast must be drawn between sense and reference. The reference of a complex expression does not contain the references of its constituent parts. Thus, to take the example of singular terms first, the value of a function for an argument does not contain the argument as its part. The number 1 does not contain the number $\pi/2$ despite being the value of the function *sin* for $\pi/2$. Similarly for sentences: the truth value True does not contain either 2 or 3 or *less than*. But if we consider the *sense* of the sentence, rather than its reference, things will look differently: the thought expressed by the sentence "2 < 3" *does* contain the senses of "2", "3" and "<". Likewise, the sense of the expression "$sin(\pi/2)$" contains the senses of "*sin*" and "$\pi/2$".

The same is true of relational complexes, facts and truth conditions. The truth condition of "$2 < 3$" is supposed to be different from the truth condition of "The sun is hot" because the former contains 3 as its part, whereas the latter does not. There is a difference between the truth condition and the sense of a sentence: the former contains the reference, rather than the sense, of each of its constituent parts. This point, however, does not affect the matter at hand. It is enough that truth conditions are wholes composed of parts.

Russell's argument concerns asymmetrical relations. For the sake of simplicity I will present it for binary relations only; its extension to relations with a greater number of arguments presents no difficulties. All binary relations can be divided into two groups: symmetrical and asymmetrical. A relation R is symmetrical iff, for all x and y, if x is in relation R with y, then y is in relation R with x. Otherwise, R is asymmetrical. Asymmetrical relations can be further divided into strongly asymmetrical and weakly asymmetrical ones. An asymmetrical relation R is strongly asymmetrical iff, for all x and y, if x is in R with y, then y is *not* in R with x. Otherwise, R is weakly asymmetrical.

The argument is based on an observation of John Stuart Mill. He remarked in *The System of Logic* that "to predicate of A that he is the father of B, and of B that he is the son of A, is to assert one and the same fact in different words. The paternity of A and the filiation of B are not two facts, but two modes of expressing the same fact".[1] Although the observation is Mill's, it was Russell who perceived the difficulty it

leads to.

In *The Principles of Mathematics*, Russell held that for any relation R there is another relation, denoted by "R^{-1}", called the inverse of R, such that for any x and y, xRy holds if and only if $yR^{-1}x$ holds. If R is a symmetrical relation, then R and R^{-1} are one and the same relation.[2] Otherwise, R and R^{-1} are two different, though closely connected, relations. The connection consists in the fact that whenever xRy holds, $yR^{-1}x$ holds, too. In subsequent writings Russell rejected this theory, because it implied that there are two different relational complexes – ARB and $BR^{-1}A$ – where a simple sense of reality tells us that there is only one, albeit expressed in two different ways. In *The Theory of Knowledge*, Russell wrote:

> as far as we are dealing only with the external fact in virtue of which the proposition that A precedes B is true, it seems obvious that it consists of two events A and B, occurring one after the other, and that it is a purely linguistic matter whether we decide do describe this fact by saying that "A precedes B" or that "B follows A".[3]

The conclusion Russell drew was that there are not two relations, "precedes" and "follows", but only one relation, whose obtaining or failing to obtain can be expressed in different ways, depending on the order in which we mention the relata. "A precedes B" and "B follows A" express the same fact (or, in other words, the existence of the same relational complex).

This seemingly obvious observation leads to unexpected difficulties. For example, if we follow Barwise and Etchemendy[4] in representing the relational complex expressed by the sentence "A is in the relation R to B" by the ordered triple <R, A, B> we will face a difficulty. The triple <R, A, B> is certainly different from the triple <R^{-1}, B, A>, even though the corresponding fact, of which they are supposed to be models, is only one. It would not be a good defence to say that the model was not intended to be perfectly faithful to reality and never to introduce any artefacts. The simple answer is that once we notice a flaw in the model, we should try to amend it. Neither can we substitute a set for the ordered triple, since $\{A, B, R\}$ is still different from $\{A, B, R^{-1}\}$. Moreover, the two unquestionably different facts – ARB and BRA – would be represented by the same set, viz. $\{A, B, R\}$.

The case of truth conditions is basically the same. The truth condition of the sentence "A precedes B" must somehow involve A, B and the relation *precedes*. The truth condition of the sentence "B follows A"

must likewise involve A, B and the relation *follows*. However, a simple sense of reality demands that the truth condition of "A precedes B" be exactly the same as the truth condition of "B follows A". We must find a way of construing truth conditions that would accommodate this observation.

Once we concede that "ARB" and "$BR^{-1}A$" assert one and the same fact, then, as I have just shown, we have to admit that the difference between R and R^{-1} is merely verbal. However, for the same reason we also have to admit that the order in which we choose to put the relata is but a way of asserting the fact, and not a feature of the fact itself. In other words, we have to admit that the difference between the two orders of the relata is an artefact of the model and not a property of reality itself. No real property of the asserted fact corresponds either to the difference between R and R^{-1}, or to the difference between the two orders of the relata.

We might put the argument in this way: the fact that A *precedes* B is a complex whole, which consists of A, B and some relation, all put together in a certain way. We inquire whether the order of the relata and the difference between *precedes* and *follows* are real properties of the fact, or whether one, or both, belong merely to our way of expressing the fact. Neither can be real, however, because if we change both the order of the relata and the relation, we obtain the same composite whole. If there were a real difference between R and R^{-1}, and a real difference between the two orders of relata, then $BR^{-1}A$ would have to differ *doubly* from ARB. One substitution cannot cancel the other, i.e., the effect of substituting one order of relata for the other cannot cancel the effect of substituting one relation for the other, if both the order and the relation are parts of the fact, and not just external traits introduced by the way of speaking. This seems undeniable. Suppose I have grey hair and a long beard. If I now dye my hair jet black and shave off my beard I am bound to look different: shaving the beard cannot cancel the effect of dying the hair. Similarly with relational complexes: the substitution of one central relation for another cannot cancel the effect of substituting one order of the relata for another – assuming, that is, that both differences are real.

Since we write in space and speak in time, we cannot but mention one of the relata before the other. This does not mean, however, that that relatum is "first" in reality. Within the fact itself, neither of the two members is discriminated against the other. To repeat: because "A is

in the relation R to B" and "B is in the relation R^{-1} to A" assert one and the same fact, neither the order of the members, nor the difference between R and R^{-1} can correspond to anything real in the world.

Having reached this conclusion we must now face the question: what distinguishes the fact that A precedes B from the fact that A follows B? We have agreed that the central relation constituting the two undoubtedly different, and even incompatible facts, is the same. We have also concluded that they do not differ by virtue of a difference in the order of their members, contrary to what might be suggested by an equivalent way of asserting the second fact, namely "B precedes A". And the two facts are obviously not different because of different relata.

The solution proposed by Russell in *The Theory of Knowledge* is as follows. We assume the existence of two special relations K_R and L_R for each relation R. Now, the difference in question is explained by saying that, in one case – say, when A precedes B – A is in the relation K_R to a certain object, while B is in the relation L_R to that same object. In the other case – when B precedes A – A stands in the relation L_R and B in the relation K_R, to a certain object. It remains to be decided what this object is; a point on which Russell is obscure. We can immediately rule out the supposition that (a) it is the relation R itself. Each member can occur in several complexes with R central. Now, suppose that A precedes B and that C precedes A. In that case A would have to stand to R in both K_R and L_R. But then we would not be able to explain the difference between a situation in which CRB, BRA and ARD hold, and one in which CRB, BRA and ARD hold. For consider the former case first: A stands in K_R to R because ARD holds. A also stands in L_R to R because BRA holds. B stands to R in K_R because of BRA, and in L_R because of CRB. Hence, both A and B stand in both relations K_R and L_R to the relation R. We can establish similarly the same conclusion for the latter case, i.e. when CRB, BRA and ARD hold. Hence the two cases cannot be distinguished.

We must say, therefore, that the relations K_R and L_R do not hold between the elements A and B and the relation R itself, but (b) between the elements and the whole relational complex. (b1) However, this relational complex cannot be the original complex, i.e. ARB, because then the difference between ARB and BRA would still be left without explanation. The two complexes – ARB and BRA – are different independently of any relations in which their members – i.e. A and B – may stand to other objects. The difference between ARB and BRA

is primary with respect to the fact that $(ARB)K_RA$ holds, because $(ARB)K_RA$ contains the complex ARB – different from BRA – as its constiuent. Furthermore (and this, I think is the decisive argument against the present proposal), in case (b1), we would have not just two, but four possibilities, two for each of the two complexes ARB and BRA:

(1) $(ARB)K_RA$ and $(ARB)L_RB$
(2) $(ARB)L_RA$ and $(ARB)K_RB$
(3) $(BRA)K_RA$ and $(BRA)L_RB$
(4) $(BRA)L_RA$ and $(BRA)K_RB$

It is no answer to say that two of these complexes are impossible. Whatever the meaning of "impossible" might be, to leave this impossibility without any explanation would be no better than to leave the very difference between ARB and BRA unexplained.

We are led to conclude that (b2) the relations K_R and L_R, which were invoked in order to explain the difference between the complexes ARB and BRA, are borne by the elements A and B to some other complex or complexes. However, it must be the same complex in both cases, that is, for both ARB and for BRA. (We designate that unique complex by "$S_R\{A, B\}$".) Otherwise we would again have four possibilities, instead of two, as shown above.

We are presently considering a theory in which the fact that A is in the relation R to B should be analyzed as a conjunction of three facts: $S_R\{A, B\}$, $S_R\{A, B\}K_RA$ and $S_R\{A, B\}L_RB$. The fact that BRA holds consists in the following three facts: $S_R\{A, B\}$, $S_R\{A, B\}L_RA$ and $S_R\{A, B\}K_RB$. The relation S_R is symmetrical.

Russell held that the relations K_R and L_R are asymmetrical. It might be thought that this would lead to a vicious circle: to analyze the fact that a certain asymmetrical relation holds we would seemingly have to assume that another asymmetrical relation holds. This is not a vicious circle, however. Our problem was to analyze the difference between two complexes which can be obtained from a binary asymmetrical relation and two elements, such that one can be transformed into the other by swapping the elements. For example, we might be concerned with the difference between the complex A-precedes-B and the complex B-precedes-A. Despite the fact that the relations K_R and L_R are

asymmetrical, we cannot form two complexes in each case, since the swapping of elements will not generate a consistent complex. Russell justifies this claim by saying that a relational complex is an object of a fundamentally different category than a simple object, and if a place in a relational complex (of a higher category) admits a complex, it cannot admit a simple object, and vice versa. Now, because the relation K_R (and the same applies to L_R) joins a complex and a simple object, we cannot swap the members of a complex with K_R as the central relation. We may express the same on the level of language by saying that in the expression "$S_R\{A, B\}K_RA$" the substitution "$S_R\{A, B\}$"/"A"; "A"/"$S_R\{A, B\}$" is not a substitution *salva significatione.*

However, instead of thus invoking a kind of theory of grammatical categories, we might assume that all objects, simple as well as complex, are substitutable salva significatione in all complexes. In order to avoid circularity in the analysis of asymmetrical relations we might simply assume that the relations K_R and L_R are symmetrical. This leaves the analysis basically unaltered, though it is a departure from Russell's views, in which the doctrine of grammatical and ontological categories played an important role, and was not an ad hoc device introduced merely to deal with asymmetrical relations.

The following theory can be reconstructed from Russell's *Theory of Knowledge.* Asymmetrical relations can be divided into homogeneous and heterogeneous relations. A heterogeneous relation connects elements of different ontological categories and gives rise to heterogeneous complexes. These complexes present no difficulties. Homogeneous asymmetrical relations, i.e., those that connect elements of the same ontological categories, do not really exist. The fact asserted by the sentence "A is in the relation R with B", where R is supposed to signify such a relation, is only apparently an atomic complex consisting of the central relation R and two objects A and B. In reality it is a molecular fact, a conjunction of three facts, one of which is symmetrical and two which are both asymmetrical and heterogeneous. Each of the latter contain one of the two relations K_R and L_R as the central relation, and two members: one of the two objects A and B and a certain symmetrical complex, consisting of both A and B and a symmetrical relation S_R. In this way homogeneous asymmetrical relations are eliminated in favour of symmetrical and heterogeneous asymmetrical relations.

This theory has some rather counterintuitive consequences. We might ask what the symmetrical relation S which serves as a basis for an

asymmetrical complex really is. Clearly, in the case of the asymmetrical relation *is earlier than* it will be the relation *are together in time*. In this way, the fact that Caesar was born before Napoleon is reduced to the conjunction of three facts: (1) the birth of Caesar and the birth of Napoleon both belong to one history; (2) the birth of Caesar occupies a preceding position in the complex asserted in (1); (3) the birth of Napoleon occupies a succeeding position in the same complex. The fact that the birth of Caesar and the birth of Napoleon are related in time is primary and atomic; the fact that Caesar was born before Napoleon is secondary and is founded upon that previous fact. Common sense would rather suggest the opposite. The fact that the birth of Caesar and the birth of Napoleon are related in time seems rather to be merely an alternation of two facts: that one is before the other, or the other is before the one (assuming the relation *before* is reflexive). It is the fact that Caesar was born before Napoleon that ought to be atomic, or at least simpler, than the fact that Caesar was born earlier or later than Napoleon.

The situation has the characteristics of a mild paradox. On the one hand we tend to believe that the fact of A's being earlier than B is identical with the fact of B's being later than A. On the other hand, we have a feeling that the asymmetrical relation of time order is primary with respect to the symmetrical relation, which can be defined as a suitable alternation. However, one of these intuitions must be foregone.

If we are willing to follow Mill and Russell and part with the intuition that asymmetrical relations are real, we will have to admit that the set theoretical model of relations is quite inadequate. That is because for every relation R, understood set-theoretically, there exists the relation R^{-1} such that for all x and y $(x, y) \in R$ iff $(y, x) \in R^{-1}$. Treated as sets of pairs, if R is asymmetrical R and R^{-1} are two different relations. This property of the mathematical model conflicts with the conclusions we have reached concerning real relations, as opposed to sets of ordered pairs.[5]

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NOTES

[1] Mill, J. S.: *System of Logic*, p. 55.
[2] Russell, B.: 1943, *The Principle of Mathematics*, W. W. Norton and Co., New York, p. 96.
[3] Russell, B.: *The Theory of Knowledge*, p. 85.
[4] Barwise, J. and Etchemendy, J.: 1987, *The Liar: An Essay on Truth and Circularity*, Oxford University Press, New York.
[5] I am indebted to Tomek Placek, whose valuable suggestion helped me to improve the article.

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TYPES OF PREDICATION*

INTRODUCTION

The philosophical systems, in which one speaks about ontologically varied objects, in which – if they are not contradictory – some formal ontology occurs in a hidden way, are an inspiration, like natural language, for seeking such formal constructions in which some distinctions among objects are possible. There is a need to capture the main semantic idea which underlies research of this kind.

We deal below with an expression of this idea, taking into consideration its possible realizations. We propose a scheme of such a semantics and indicate some partial realization of it – which have occurred in the history of philosophy and logic.

1. REFERENCE AND COREFERENCE

We assume that there are two types of objects, to which non-empty names and predicates refer. We use here a notion of an object in the most general sense of the term, as something we can speak or think, without prejudging its existence or the manner of its existing (Twardowski, Meinong, Husserl).

We distinguish two kinds of reference relations of names (predicates) to objects: *direct* and *indirect reference.* Direct referring we call for short *reference* and indirect reference we call *coreference.*

2. REFERENCE AND DENOTATION: COREFERENCE AND CONNOTATION

The pair *reference-coreference* corresponds with Mill's terms *denotation-connotation.* In Mill's theory (*Systems of Logic*, Vol. I, Chapter II) some names (terms), besides having the function of denotation, referring to the objects whose names they are, also have the function of connotation, the function of referring to the properties which every

Jan Woleński (ed.), Philosophical Logic in Poland, 307–319.

object has that is denoted by this name. Mill, talking about denotation, had in mind a definite kind of reference: empirical reference.

The idea developed here has *a priori* a schematic character: the references of names are objects that belong to a distinguished universe, which is most often called 'the real world'. The main difference between two expressions resolves into the difference between the connotation and the coreference. Connotation was connected with the property (set of properties) of the denotata bearing this name; here the coreference of the given name is the given object (or objects). The coreference of the given name occurs in the context of other objects of that kind, belonging to the same ontological order – however different than the one to which its reference belongs.

3. SUBCATEGORIES OF THE NAME CATEGORY

All names can be divided semantically into: possessing (or lacking) reference and possessing (or lacking) coreference. These four subcategories of the name category (n) are as follows:

(rc) – *referential-coreferential names,*
($r\bar{c}$) – *referential-non-coreferential names (exclusively referential),*
($\bar{r}c$) – *non-referential-coreferential names (exclusively coreferential)* and
($\bar{r}\bar{c}$) – *non-referential-non-coreferential names* (empty).

The first two categories of names fall into the *referential* name category ($rc \subset r, r\bar{c} \subset r$) and the remaining two are included in the *non-referential name category* ($\bar{r}c \subset \bar{r}, \bar{r}\bar{c} \subset \bar{r}$). Moreover, we will divide names into *singular (i)* and *general (g)*. Finally, we have the following subcategories of the name category: $irc, grc, ir\bar{c}, gr\bar{c}, i\bar{r}c, g\bar{r}c, i\bar{r}\bar{c}$ and $g\bar{r}\bar{c}$.

An example of a general referential-coreferential name (grc) is *man*: it directly refers to individuals, such as Socrates, Plato, the author of Hamlet, and indirectly to objects of another kind than man as a human being (in the expressions: man as an idea, man as a universal, man as a concept, man as a class).

Definite descriptions are singular referential-coreferential names (irc). The name *the highest mountain in Europe* has reference (in common

with the proper name *Mont Blanc*) and coreference – the highest mountain in Europe as an object in another ontological order (for instance singular concept, singular class).

Proper names occurring in a natural language are singular referential names. Because of their conventional use, arbitrary in the choice of bearer, they do not as such have a coreference. Users of a language can connect however a certain coreference with a given proper name, which they usually express by the sentential construction of the type A *est* B, where A is a proper name and B is a referential-coreferential name (definite description or universal name). Thus in the sentence *Socrates is the teacher of Plato* we assign to the proper name *Socrates*, having the common reference with the description *the teacher of Plato*, the coreference – the teacher of Plato, treated for instance as a singular concept, included into a certain hierarchy of objects (relation of subsumption) ontologically different from the reference of this name.

Adjectival names, such as *white*, *green*, *heavy*, *sharp*, are general referential names in virtue of whose contextuality and indistinctness we do not ascribe a priori (like proper names) a coreference. However, we can talk about coreference in the case of compound names, formed of adjective names and universal names which are indefinite descriptions. For instance, a name such as *white man* already possesses the coreference – a white man in general.

In a natural language we also find singular non-referential (merely coreferential) names ($i\bar{r}c$) such as *Zeus* or *Aphrodite* and thus also general merely coreferential names ($g\bar{r}c$), such as *God of Olympus*. They refer indirectly to certain objects (to intentional objects – in one interpretation).

In a natural language one can construct also *quasi*-descriptions formed of referential names according to the pattern of indefinite or definite descriptions: they are the empty names. Examples of the first would be: *son of childless woman* or *rectangular circle*. The others can be represented by: *the man who constructed the solar system*. An empty name is also a negation of a maximally general name, such as *being*.

4. WIDE AND NARROW UNDERSTANDING OF THE NAME CATEGORY

Logical languages can be divided according to the name categories they admit into two types: languages with a *wide* understanding of the name category, including singular names as well as general ones, and languages with a *narrow* understanding of the name category, allowing singular names alone.[1]

Languages of the first type express a structure of elementary sentences by the word *is*, which is a sentence-forming functor for two name arguments. Languages of the second type allow for at most one counterpart of the word *is*, *is-the-same-object-as*, the identity functor. On the basis of the first type of language name calculi are constructed, and the predicate calculi are the constructions of the second type of language, where predicates substitute for general (referential and non-referential) names.

5. PREDICATION

Predication consists in the assignment of coreference of a given name or – in case of languages with a wide understanding of the name category – ascription of reference of this name to another name, to a general and referential name.

5.1. *Wide understanding of the name category*

Predication in languages with a wide understanding of the name category is realized by sentential constructions of the type A *est* B. By constructions of this type we state that that which is a reference of the name A is a reference of the name B, and that the coreference of the name B is also coreference of the name A. Moreover, if A is a singular name or a general referential-coreferential one, and B is a singular referential-coreferential (definite description) or a general name, then the names A and B of this sentence construction are both in the relation of predication.

5.2. *Individual and general names*

Considering predication we find, in the traditional logic, a division of names into *individual* and *general*. Individual names may occur only as subjects in constructions of the type *A est B*, general ones are suitable for predicates. In a class of referential names some would be the proper names, and the definite descriptions or general (as contrasted with singular) names.[2]

5.3. *Narrow understanding of the name category*

The traditional semantic syntactical conception of propositions take as a canon the subject-predicate structure of elementary sentences expressed by the scheme *A est B*. The new conception, proposed by Gottlob Frege as the basic one, takes the structure φA, where the predicate φ corresponds with the expression *est B* from the traditional scheme and A is a singular name. Assuming that references of monadic-predicates are functions or classes, we ascribe a certain property to the reference of A in the predication scheme φA. This property we can treat as the coreference of this predicate.

Frege himself underlined the fact of such a specification of the reference of the given name A by ascribing it this but no other property, the *mode of presentation* (*Art des Gegebenseins*), which he called its *sense*. The sense thus understood we are inclined to bind with the predicate as its coreference (by the other interpretation of coreference of predicates), which we ascribe to the singular object in predication.

In the language of the theory predication of singular objects (individuals) has a form $A \in \alpha$, where the reference of A is an individual, whereas α is a set (class) name. We can treat class names as singular names indirectly referring to classes as certain abstract objects. Class names would be in this interpretation the singular exclusively coreferential ones ($i\overline{r}c$). Predication lies here in assigning coreference to the given individual – the class, to which it belongs.[3]

6. FOUR TYPES OF PREDICATION

Taking into account the object of predication, we will distinguish between *referential* and *coreferential predication*. In the first case only objects belonging to the distinguished universe are the objects of predi-

cation. In the other case objects of predication are the objects belonging to the second ontological order. According to the above, we can divide logical languages with referential predication into two types: languages *with coreferential predication* and *without coreferential predication.* Combining the above division with the previous division of logical languages into languages with wide and narrow understanding of name category, we obtain four types of language, and as a result, four types of predication:

(1) *referential-coreferential predication with a wide understanding of the name category,*
(2) *exclusively referential predication with a wide understanding of the name category,*
(3) *exclusively referential predication with a narrow understanding of the name category* and
(4) *referential-coreferential predication with a narrow understanding of the name category.*

6.1. *First type of predication*

Predication of this kind we already have in Plato's theory of ideas, where general names, such as *human being*, *animal*, have references, objects accessible to the senses, which are merely temporal bearers of these names, and besides they have coreferences: they refer (indirectly) to timeless objects, to the human being idea and to the animal idea, respectively.

The classical theory of predication

Predication of this type is finally the subject of the Aristotle's classical predication theory. Names (terms) are divided here into singular and general (*De int.*, VII, 17a39).

In sentences of subject-predicate form the former are capable of being subjects only, the predicates can be only the general names (*Cat.*, II); those which are names of the second substances are also capable of being subjects (*Cat.*, III, V).

Singular names are the names of individuals, of first substances (cf. *Cat.*, V, 2a11), which we will treat here as referential singular names (ir). Next, the general names (the names of universals) are the names

of second substances and they are capable of being predicated of particular individuals (*Cat.*, V, 2a37) – they are in this way the general referential-coreferential names (grc) – referring directly to the things (first substances) and indirectly to the second substances, understood also as essences of things.

In the language of Aristotle's logical works the names which occur in subject-predicate sentences are referential ones – the categories of singular referential and general referential names coincide respectively with the singular name category ($ir = i$) and the general name category ($gr = g$). According to classical predication theory, in a sentence of type *A est B*, A is a singular name (i) or an universal one (of the category u: $u = grc$, $u \subset g$), B is a general name (g), and the word *est* has the category $o\backslash s/g$ (s – the sentence category, o – the substance category: $i \subset o, u \subset o$). The propositions *Socrates est homo* (with the categorial index $i(o\backslash s/g)u$) and *homo est animal* ($u(o\backslash s/g)u$) are examples of meaningful sentences according to this theory.[4] The syllogistic, developed by Aristotle in *Analytica priora*, can be treated as the syntactical development of a certain fragment of this theory.[5]

We meet also with the conceptualistic interpretation of classical predication theory (Avicenna, Ockham), in which the coreferences of universal names are the concepts. Ockham distinguishes (*Summa logicae*, XII) between the names which are names of both things and concepts (*first intentions*), and names which refer exclusively to the concepts of these concepts (*second intentions*). We would here interpret the names of first intentions as general referential-coreferential names (grc) and the names of second intentions as general exclusively coreferential ones ($g\bar{r}c$).

Logical constructions

Examples of contemporary logical constructions which we can treat as partial syntactical realizations of the first predication type are: the system introduced in one of Lejewski's papers[6] and the calculus constructed by Iwanuś.[7] The first of these constructions inferentially includes Leśniewski's ontology (his standard fragment from 1930), while the other one includes a certain fragment of elementary ontology;[8] every name in this calculus has reference or coreference and is an exclusively referential or exclusively coreferential one.[9]

6.2. *Second type of predication*

We meet predication of the second type, exclusively referential on a wide understanding of a name category, in Kotarbiński's reism, where direct reference concerns things, and indirect reference, appealing for instance to universals, is treated as a metaphorical form of expression. Propositions treating relations between the coreferences of universal names are replaced, according to the semantical aspect of this doctrine, by propositions expressing corresponding relations between the references of these names.

The ontology of Leśniewski is an example of a name calculus which is a formulation of this predication theory, preferring nominalistic philosophical constructions.[10] On the basis of this logical system one type of predication is realized: referential predication. All names are divided into referential and non-referential, the latter being identified with the empty names. Referential names divide, as in the first predication type, into singular and general.

The earlier analyzed proposition *Socrates est homo* $(i(n\backslash s/n)g)$ is a meaningful sentence within the framework of this predication doctrine $(i \subset n,\ g \subset n)$. In the above example, the word *est* occurs in an identical meaning as the ontological epsilon functor. Next, in the second of analyzed propositions *homo est animal* the word *est* occurs in another meaning as in the sentence *Socrates est homo* (*est* = ε); the meaning of this word we express by the functor of (weak) inclusion, as defined by: $\Pi x(x\varepsilon homo \rightarrow x\varepsilon animal)$.[11]

6.3. *Third type of predication*

The third predication type is the realization of Frege's idea, the restriction of the name category to singular names. Here we work on the basis of the *classical predicate calculus*, in which *de facto* they are reduced to the singular referential names $(ir = i)$.[12]

The example *Socrates est homo* would be interpreted here as: *Socrates homo-est* (more: *homo-est Socrates* – $(s/i)i$), and next *homo est animal* will be transformed into: $\Pi x(x\ homo\!-\!est \rightarrow x\ animal\!-\!est)$.[13]

6.4. *Fourth type of predication*

The last of the distinguished predication types is the referential-coreferential predication with a narrow understanding of the name category.

The conception of sense and reference

Frege's theory of sense and reference represents this type of predication. The reference of a name, its *Bedeutung* is the singular object which bears it; the equivalent of its coreference would be its sense (*Sinn*).[14]

For names of objects Frege uses the term *proper name (Eigenname)*.[15] The reference of a predicate is a function according to Frege. The reference of monadic predicate (*Bedeutung des Begriffwortes*) is a concept (*Begriff*), considered as a monadic function whose values are the two truth-values.

A similar role to the pair *Sinn-Bedeutung* of Frege's theory is performed by a pair *intension-extension* in Carnap's semantics. The main difference is that the references of (singular) names and predicates in Frege are respectively singular objects and functions; in Carnap however the names denote individuals whereas the predicate extensions are classes.[16] Moreover, for Carnap a name's intension is an individual concept and the intension of a monadic predicate is a property.[17]

Logical constructions

Lejewski's *theory of non-reflexive identity (TNI)* is an example of logical construction of this predication type,[18] in which, unlike the *classical identity theory* (usually included in predicate calculus), where the reflexivity of this functor is assumed – we have the axiom: $x = y \leftrightarrow \Sigma z$ $(z = x \cdot z = y)$. The elementary proposition $x = y$ here is read: x *is-the-same-object-as* y; it is true only, when both arguments of this functor are referential (singular) names. The system has rules analogous to those of Leśniewski's ontology. The formulation of the *nominal extensionality rule* of this system does not, however, allow one to distinguish between objects, to which the coreferential names refer indirectly; i.e. intentional individuals are here non-distinguishable, in the sense that anything capable of being predicated in this language for instance about Pegasus is capable of being predicated about Zeus and vice versa.[19]

The systems of *free logic* can be considered as constructions having

also as their aim the realization of this predication type. The category of singular names includes referential as well as exclusively coreferential ones. In systems of free logic the existence predicate $E!$ is often introduced by axioms: $\varphi x \rightarrow (E!x \rightarrow \Sigma y(\varphi y))$ and $\Sigma x(\varphi x) \rightarrow \Sigma x(E!x \cdot \varphi x)$, but the identity functor (predicate) is determined by: $x = x$ and $x = y \rightarrow (\varphi x \rightarrow \varphi y)$.

K. Lambert and T. Scharle showed[20] that the enrichment of this axiom system with $\sim E!0$ (zero is the non-referential nominal constancy symbol) and with the axiom $\sim E!x \cdot \sim E!y \rightarrow x = y$ is an equivalent of a certain fragment of Lejewski's *TNI*, giving the translation rules from one system to the other.[21]

The systems of free logic and the theory of non-reflexive identity explicate only the assumed existence notion, which is axiomatically or definitionally (Lejewski) introduced. The languages of these systems are not essentially richer in describing the world of intentional objects than classical predicate theory, with the exception that one can express truthfully here for instance *Pegasus does not exist* – having in mind only, that the name *Pegasus* has no reference, which was expressible but false in the classical theory.

A similar construction, in which the predicate of existence occurs explicitly, is presented by J. T. Kearns.[22] Kearns proposes to distinguish *strong* and *weak* predicates. Every predicate φ has a strong φ^+ and weak φ^- equivalent, which are introduced by the definitions: $\varphi^+x \leftrightarrow Exists(x) \cdot \varphi x$ and $\varphi^-x \leftrightarrow (Exists(x) \rightarrow \varphi x)$. This construction does not obliterate the differences among intentional individuals.[23]

7. CONCLUSIONS

The logical constructions representing the fourth predication type ought not only to explicate one kind of existence for the referential names, but also ought to explicitly introduce other existence predicate, whose argument would belong to the coreferential name category.

Especially interesting would be these logical constructions, which independently of the predication type realized, operating with two predicates (functors) of existence, would have laws (rules) of extensionality in the frame of each kind of individual (object), without obliterating in this way the differences between individuals which do not have the first kind of existence, and especially would not identify the latter with the

standard assumed contradictory object; empty (contradictory) names ought to be distinguished from exclusively coreferential names.

The need to talk about objects other than for instance concrete ones (really existing objects) is, we consider, strong enough to inspire logical investigations in this field.

One can point to certain ontological distinctions occurring in contemporary philosophy,[24] for which the following logical distinctions would be helpful: *object–representation* (Twardowski),[25] *object–noema* (Husserl); *real object–intentional object*, functioning in different contexts, for instance in the work of art theory (Ingarden); *the bearer of value–value* (Scheler), etc.

Metaphysical solutions to the question what there is do not belong to logic. The logic would enable us, however, to talk about different objects, giving philosophers the proper instruments for their theories. The logicians can do it without ontological commitment; they can do as mathematicians do in creating the theories of possible worlds. They could, however, express their ontological commitment more delicately, like Kronecker for instance, having the theories of natural, real and complex numbers, he believed that only the first describes the real world – *Die ganzen Zahlen hat Gott geschaffen, alles andere ist Menschenwerk.*

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NOTES

* This is an English version of my paper: 1988, 'Cztery typy predykacji' (Four types of predication), *Studia filozoficzne* **6–7**, 117–126.

[1] C. Lejewski distinguishes in his works three classical types of logical languages: Aristotelian language (the name category forms here referential names), Frege-Russellian language (singular referential names) and Leśniewski's language (referential names – general and singular, and non-referential – empty names). See for instance: Lejewski, C.: 1965, 'A Theory of Non-Reflexive Identity and Its Ontological Ramifications'. In: Weingartner, P. (Ed.) *Grundfragen der Wissenschaften und ihre Wurzeln in der Metaphysik*, Salzburg-München, pp. 65–102.

[2] Kotarbiński, T.: 1961, 'Z zagadnień klasyfikacji nazw'. In: Kotarbiński, T., *Elementy teorii poznania, logiki formalnej i metodologii nauk*, 2nd Ed., Ossolineum. English: 'On the classification of names'. In: Kotarbiński, T., *Gnosiology*, Pergamon Press, Oxford, 1966. The notion of individual name depends on an assumed individual theory.

[3] Quine in his logical system treats names such as *clever, town* as singular ones – they refer (indirectly according to us) to classes. See Quine, W. V. O.: 1955, *Mathematical Logic*, 2nd Ed., Cambridge, Massachusetts, §22.

[4] Cf. Ajdukiewicz, K.: 1934, 'W sprawie uniwersaliów', *Przeglad Filozoficzny* **37**, 219–234. English: 'On the Problem of Universals'. In: K. Ajdukiewicz, *The Scientific World-Perspective and Other Essays 1931–1963*, Ed. by Jerzy Giedymin, Reidel, Dordrecht, 1978, pp. 95–110. The distinguishing of these name subcategories is especially important for languages using articles. Let us take for example the propositions: *Socrates is a man, Socrates is the husband of Xanthippe* and *the lion is an animal.* The first and third are equivalents to the above analyzed propositions. Ascribing to the word *is* the category $n\backslash s/n$, the definite article as functor would be of category n/o, the indefinite article of category n/u.

[5] As is known, Aristotle in his syllogistic restricted the names to the general referential ones. See Łukasiewicz, J.: 1951, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford, 4f.

[6] Lejewski, C.: 1978, 'Idealization of Ordinary Language for the Purposes of Logic'. In: D. T. Alleton, E. Carney, and D. Holdcroft (Eds.) *Function and Context in Linguistic Analysis*, Cambridge, pp. 94–110.

[7] Iwanuś, B.: 1976, 'W sprawie tzw. nazw pustych' *Acta Universitas Wratislaviensis* **290** (Logika 5), 73–90.

[8] By *elementary ontology* we mean the fragment of ontology in which the quantifiers bind only nominal variables.

[9] One of theses of this calculus is: *x really-exists or x intentionally-exists.* One can show that a thesis of this calculus is also *if x really-exists, then it is not true, that x intentionally-exists.*

An attempt at a construction of a name calculus which realizes this predication type, built according to the spirit of Leśniewski's ontology, is introduced in my dissertation: *Z badań nad zastosowaniami systemów Leśniewskiego* (Investigations on applications of Leśniewski's systems), (Jagiellonian University, Cracow 1984). A fragment of this construction was used by the author in 1986: 'Zur logischen Analyse der langen Namen in der deutschen Sprache', In: *Termini-Existenz-Modalitäten* (*Philosophische Beiträge* **4**, Humboldt-Universität zu Berlin), pp. 45–57.

[10] See Küng, G.: 1963, *Ontologie und logistische Analyse der Sprache: Eine Untersuchung zur zeitgenössischen Universaliendiskussion*, Springer-Verlag, Wien, p. 84ff. (English: *Ontology and the Logistic Analysis of Language: An Enquiry into the Contemporary Views on Universals*, D. Reidel, Dordrecht, 1963). See also his: 1977, 'Nominalistische Logik heute', *Allgemeine Zeitschrift für Philosophy* **2**, 29–52.

[11] We obtain also the same result by interpreting *homo est animal* by *homo-esse est animal-esse* (*est* is here of $(s/n)\backslash s/(s/n)$ category). Cf. Küng, G.: 'Nominalistische Logik heute' …, p. 46.

[12] In the classical predicate calculus *a exists* means $\Sigma x(x = a)$, which is tersely expressed in Quine's well-known thesis: *to be is to be the value of a bound variable.*

[13] The set theoretical interpretation of the proposition *homo est animal* would be *the class of homo is included in the class of animal.*

[14] See Frege, G.: 'Über Sinn und Bedeutung'. In: G. Frege, *Funktion, Begriff, Bedeutung*, 6th Ed., Ed. by G. Patzig, Göttingen, 1986, pp. 40–65.

[15] As synonym of this term Frege used also the word *singular name (Einzelname).* See Frege, G.: 'Ausführungen über Sinn und Bedeutung'. In: G. Frege, *Nachgelassene Schriften*, Vol. I, Ed. by Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach, F. Meiner Verlag, Hamburg, 1969, pp. 128–136.

[16] See Carnap, R.: 1968, *Einführung in die symbolische Logik*, Springer-Verlag, Wien-New York, 3rd Ed., p. 40. (English: *Introduction to Symbolic Logic and Its Applications*, New York, 1958).

[17] An n-ary relation is the intension of an n-ary predicate. Ibid., p. 40, p. 90f.

[18] Lejewski, C.: 'A Theory of Non-Reflexive Identity ...'.

[19] In the framework of *TNI a does not exists* is expressed as: $\sim (a = a)$. The special case of a rule of extensionality, assumed in this system is (a): $\Pi z(z = x \leftrightarrow z = y) \leftrightarrow \Pi\varphi(\varphi x \leftrightarrow \varphi y)$. From (a) we obtain (b): $(x = x \leftrightarrow x = y) \leftrightarrow \Pi\varphi(\varphi x \leftrightarrow \varphi y)$. Considering the thesis of *TNI* (we change the non-referential name constancy symbol) (c): $\sim (x = 0)$ we obtain from (b) a thesis (d): $\sim (x = x) \rightarrow \Pi\varphi(\varphi x \leftrightarrow \varphi 0)$. From (d) we deduce (e): $\sim (x = x)\cdot \sim (y = y) \rightarrow \Pi\varphi(\varphi x \leftrightarrow \varphi y)$. We can proof an analogous thesis in the framework of Lejewski's system in 'Idealization of ordinary language ...'.

[20] Lambert, K. and Scharle, T.: 1967, 'A Translation Theorem for Two Systems of Free Logic' *Logique et Analyse* **39–40**, 328–341.

[21] Ibid., p. 332ff.

[22] Kearns, J. T.: 1968, 'A Universally Valid System of Predicate Calculus with No Existential Presuppositions' *Logique et Analyse* **41**, 367–389. This system is constructed similarly to Lejewski's *TNI*.

[23] Plus and minus are here the predicate-forming functors. We have here the theses: $\sim$ *Exists*$(x)\cdot \sim$ *Exists*$(y) \rightarrow (\varphi^+ x \leftrightarrow \varphi^+ y)$ and $\sim$ *Exists*$(x)\cdot \sim$ *Exists*$(y) \rightarrow (\varphi^- x \leftrightarrow \varphi^- y)$. We can in the framework of this system deduce: $\sim$ *Exists*$(x)\cdot \sim$ *Exists*$(y) \rightarrow ((\varphi x \leftrightarrow \varphi y) \vee (\sim \varphi x \leftrightarrow \varphi y) \vee (\varphi x \leftrightarrow \varphi y))$.

[24] Certain ontological distinctions in classical philosophy were mentioned earlier, when we presented classical predication theory.

[25] The term *object* occurs here in the narrower meaning of presented object. The pair *object–presentation* corresponds with the pair *real object–presented object*, by a wider understanding of 'object'. See Twardowski, K.: 1894, *Zur Lehre vom Inhalt und Gegenstand der Vorstellungen*, Hölder, Wien, §4.

JAN WOLEŃSKI

REMARKS ON EXTENSIONALITY AND INTENSIONALITY

This paper deals only with extensionality and intensionality in languages modelled by propositional cacluli, i.e. languages (I shall call them "sentential languages") consisting of items built by means of functors that form sentences of sentences. Now a standard way of defining extensionality and intensionality is this. Let L be a sentential language. A context P of L is extensional if and only if its logical value (truth or falsity) is a function of logical values of its constituents. Then a sentential functor is extensional if and only if it forms an extensional context. A context (functor) of L is intensional if and only if it is not extensional.

Extensional contexts have nice properties. In particular, they obey the so-called principle of extensionality (P, Q-arbitrary sentential formulas)

$$(1) \qquad \frac{P \Leftrightarrow Q}{H(P) \Leftrightarrow H(Q)}$$

On the other hand, (1) fails in its full generality when intensional contexts are involved. This is documented by numerous examples of contexts with sentential functors like "it is known that", "it is believed that", "it is asserted that", etc., which are examples of intensional functors. Obviously,

$$(2) \qquad P \Leftrightarrow Q$$

may be true but ("BP" means "it is believed that P")

$$(3) \qquad BP \Leftrightarrow BQ$$

false, because it is perfectly conceivable that P is believed by someone, but Q is not.

Jan Woleński (ed.), Philosophical Logic in Poland, 321–325.

(1) guarantees mutual interchangeability *salva veritate* of material equivalents (m-equivalents) in extensional contexts. As (2)–(3) show, m-equivalents are not always interchangeable *salva veritate* in intensional contexts. Russell and Whitehead regarded this as "an absolute gulf" ([3], p. 666) between both kinds of contexts. Quine invented an impressive label, "opacity", to capture the non-preservation of truth under interchangeability of m-equivalents; "transparency" is, after Russell, a label for the opposite phenomenon. Thus, sentential contexts are divided into transparent (extensional) and opaque (intensional).

Carnap observed ([1], pp. 48, 54) that this classification is too simplistic. For him:

> The term 'intensional' [...] will be used not, as is sometimes done, as synonymous with 'nonextensional', but in a narrower sense, namely, in those cases in which the condition of extensionality is not fulfilled but the analogical condition with respect to intension is fulfilled. [1], p. 48.

A context P is, roughly speaking (I omit here several details of Carnap's approach) extensional if it satisfies (1); it is nonextensional if it does not. Now a context P is intensional if and only if it is not extensional and its logical value is preserved under mutual interchangeability of logical equivalents (l-equivalents). This last condition may be formally expressed by

$$(4) \qquad \frac{\vdash P \Leftrightarrow Q}{H(P) \Leftrightarrow H(Q)}$$

Intensional contexts in Carnap's sense can be illustrated by alethic, particularly logical, or deontic modalities. In fact, if P is necessary (possible) or obligatory (permiited) and P is logically equivalent with Q, then Q is also necessary (possible) or obligatory (permitted). Obviously, material equivalence does not preserve modal or deontic status.

Contexts which are neither extensional nor intensional belong to the third group distinguished by Carnap. Sentences about the so-called propositional attitudes, in particular belief-sentences (compare (2)–(3) above), are typical examples here. Even (4) is too weak to guarantee interchangeability *salva veritate* in such contexts. Carnap, in order to resolve this question, proposed a stronger kind of equivalence defined in terms of the intensional isomorphism. Since this solution raised many objections, I do not enter into details of Carnap's proposal.

Carnap's observation that the extensionality/intensionality distinction induces not two but three distinct cases is certainly valuable. However,

his quoted terminological explanation may be confusing. Note that since (4) implies (1) but not conversely, if a context P is not extensional, it is thereby not intensional in Carnap's sense. So it is sufficient to distinguish three cases: (a) that in which (1) is satisfied; (b) that in which (4) is satisfied; (c) that in which (1) is not satisfied. The succession (a)–(c) indicates that we encounter various degrees or amounts of extensionality, respectively, if someone prefers, intensionality. We can perhaps speak about strong extensionality (case (a)), weak extensionality (case (b)) and intensionality (case (c)), or, respectively, extensionality, weak intensionality and strong intensionality. Though the tradition, which groups modal (alethic), deontic and epistemic sentences into one intensional variety, supports the second way of speaking, I prefer the former one. So I contrast strong extensionality, weak extensionality and intensionality. I would like to offer another characterization of particular kinds of contexts.

Interchangeability can be considered as an operation performed on sentences. Formally speaking, we think on interchangeability as a mapping i: $L \rightarrow L$ such that if H is a sentence in which P occurs, $iH = H(P/Q)$ if and only if P is replaced by Q. Now we define

(9) $H(P)$ is strongly extensional if and only if $iH(P) \Leftrightarrow H(P/Q)$ for any Q which is m-equivalent with P.

(10) $H(P)$ is weakly extensional if and only if $iH(P) \Leftrightarrow H(P/Q)$ for any Q which is l-equivalent with P.

(11) $H(P)$ is extensional if and only if it $H(P)$ is strongly or weakly extensional.

(12) $H(P)$ is intensional if and only if $\neg(iH(P) \Leftrightarrow H(P/Q))$ for some Q which is equivalent with P.

These definitions show that interchangeability is related to rules of inference and then to consequence-operations. We can say that interchangeabililty in (9) is induced by the standard operation Cn used in formalizations of sentential calculus. Let us say that a consequence operation is faithful with respect to a discourse D if and only if it never produces false conclusions from true premises. Thus, if D is a strongly extensional discourse, Cn is D-faithful it and conversely. This observation leads to other account of strong extensionality. On the other hand,

Cn is not faithful with respect to a discourse in which modal or deontic sentences occur. However, we can extend Cn to a new consequence, say Cn^*, by adding further axioms, in particular a counterpart of (4). Although there are controversies as to which system of modal or deontic logic is the "correct" one, Cn^* certainly records some important intuitions and expectactions when interchangeability salva veritate in weakly extensional contexts is involved. Unfortunately, nobody knows exactly what should be done to extend Cn to a consequence-operation which would be faithful with respect to an intensional D. In particular, the concept of the intensional isomorphism does not fully help in solving this question.

Any realistic account of believing or knowing must admit that (4) may not be satisfied in related discourses. So if a logician proposes (see for instance [2]) a logic for belief-sentences closed by a consequence operation satisfying (4), he automatically decides that the intensional contexts he deals with are treated as weakly extensional. It would be difficult to regard this proposal as a realistic analysis of actual acts of believing. The same concerns epistemic logic which assumes that knowing subjects are omniscient. Now we see that there is an objective reason to group together, into the extensional variety, the contexts which I called strongly extensional and weakly extensional. Truth which is strongly extensional or weakly extensional is closed under related consequence operations. Thus, to concentrate on a difference between weak extensionality and intensionality, we have sets of alethic or deontic sentences which are deductive systems (in the metalogical sense) without a further ado, but the same does not holds for intensional contexts; if you take an arbitrary set of belief-sentences, you must something additionally say to have an epistemic deductive system.

This situation motivates different attitudes toward so-called intensional logic. Some logicians (Lesniewski, Tarski, and Quine, are examples here) consider intensional contexts as essentially defective from the logical point of view and, they are inclined to banish them from the scope of logic. On the other hand, there are logicians (Church and Montague are among them) who argue that this solution is too simple to be good, and look for a suitable system of intensional logic.

Although I am sceptical about intensional logic, it is not my intention to make a issue verdict against it. Rather I consider my remarks as an attempt to indicate where the critical point is. Simply speaking, since logics are generated by consequence operations, a real success

with intensional logic basically depends on finding a consequence operation which would be formally "natural" and faithful with respect to intensional discourse. Why does "natural" occur in quotes in the last sentence? It is because even monotonicity of an "epistemic" consequence is disputable. This shows where we are.

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BOGUSŁAW WOLNIEWICZ

ON THE SYNTHETIC A PRIORI

1. LOGICAL SYSTEMS

Take any propositional language L, i.e. any algebra of formulae, and an arbitrary consequence (i.e. closure) operation Cn on it. Following Suszko (cf. R. Suszko, D. J. Brown, S. L. Bloom "Abstract Logics", *Dissertationes Mathematicae* **102**, Warsaw 1973) any couple of the form (L, Cn) we call an *abstract logic.*

Moreover, let the set of propositions $Z_0 \subset L$ be the totality of truths expressible in the language L. Any triple of the form (L, Cn, Z_0) we call a *logical system* of the language L.

Henceforth let us assume that in our case we have:

(L1) The logical system (L, Cn, Z_0) is classical.

This is to say that in L there are certain connectives – written e.g. as $\wedge, \vee, \neg$ – which are characterized by the consequence operation Cn as classical conjunction, classical disjunction, and classical negation, respectively. It means, moreover, that the abstract logic (L, Cn) contains complete sets. Let $\underline{Z}$ be the totality of them. (We use the letter Z, as in Polish "complete" is "zupełny".) Then we have by (L1):

(1) $\alpha \in CnX \Leftrightarrow \bigwedge_{Z \in \underline{Z}}(X \subset Z \Rightarrow \alpha \in Z)$,

(2) $Z_0 \in \underline{Z}$.

2. SEMANTIC SYSTEMS

Take any quadruple of the form $((L, Cn, Z_0), SE, \underline{R}, Z)$, where the initial triple is a classical logical system, SE is an arbitrary set, $\underline{R}$ is an arbitrary collection of sets, and Z is a function of the form $Z : \underline{R} \to P(L)$, i.e. an arbitrary mapping of $\underline{R}$ into the power set $P(L)$. Let that quadruple satisfy the following two groups of conditions. The first is concerned with the collection $\underline{R}$, and it is *ontological* in character:

Jan Woleński (ed.), Philosophical Logic in Poland, 327–335.

(R1) $\underline{R} \subset P(SE)$

(R2) $\underline{R} \neq \emptyset$

(R3) $\cup\underline{R} \neq SE$

(R4) $\cap\underline{R} \neq \emptyset$

(R5) $\bigwedge_{R_1,R_2\in\underline{R}}(R_1 \subset R_2 \Rightarrow R_1 = R_2)$.

The second is concerned with the mapping Z, and it is *semantic* in character:

(Z1) $Z/\underline{R}/ \subset \underline{Z}$

(Z2) $Z_0 \in Z/\underline{R}/$

(Z3) $\bigwedge_{\alpha\in L}\bigwedge_{R\in\underline{R}}(\alpha \in Z(R) = \bigvee_{x\in R}\bigwedge_{R'\in\underline{R}}(x \in R' \Rightarrow \alpha \in Z(R')))$,

where $Z/\underline{R}/$ is the image of $\underline{R}$ in $P(L)$ under Z. A quadruple satisfying all the conditions stipulated we call a *semantic system.*

In a semantic system we call the elements of SE *elementary situations*, those of $\underline{R}$ – *realizations* (or "possible worlds"), and the collection $\underline{R}$ itself – the *logical space* of the language L. Elementary situations belonging to the union $\cup\underline{R}$ are *possible*; those belonging to the complement $SE - \cup\underline{R}$ are *impossible*; and those belonging to the intersection $\cap\underline{R}$ are *necessary*. It follows easily from conditions (R1)–(R4) that there must be at least two elementary situations, with at least one of them necessary, and at least one impossible. The set $\cup\underline{R} - \cap\underline{R}$ consists of elementary situations that are *contingent*, but it may be empty. (Obviously, the whole of that conceptual framework derives from Wittgenstein's *Tractatus*.)

By condition (Z1), to each realization R there corresponds in the language L a complete set of propositions $Z(R)$. The formula "$\alpha \in Z(R)$" is read: "proposition α is satisfied in realization R". If for a given proposition α the implication $x \in R \Rightarrow \alpha \in Z(R)R$ holds for every $R \in \underline{R}$, the elementary situation x is said to *verify* that proposition (or to be its "verifier"). Condition (Z3) makes sure that every proposition satisfied in a particular realization has a verifier in it. (This condition is highly disputable, but in the following no use will be made of it.)

3. REALITY CONDITION

Condition (Z2) makes sure that the totality of truths Z_0 contained in the language L reflects some reality. So let us call it for short the "reality condition" (for the semantic system in question). And let us consider what it means.

The totality of realizations in which a given proposition is satisfied – i.e. the set $M(\alpha) = \{R \in \underline{R} : \alpha \in Z(R)\}$ – we call the *logical locus* of that proposition (i.e. the locus determined by it in the logical space $\underline{R}$). Condition (Z2) turns out then to be equivalent to the thesis that *propositions with the same logical locus have the same truth-value.*

Indeed, take the truth-value of α to be simply the characteristic function of the set Z_0. I.e.,

$$v(\alpha) = \begin{cases} 1, & \text{if } \alpha \in Z_0 \\ 0, & \text{if } \alpha \notin Z_0 \end{cases}$$

Then our thesis takes the form:

$$(3) \qquad M(\alpha) = M(\beta) \Rightarrow v(\alpha) = v(\beta).$$

Contraposing it assume the antecedent: $v(\alpha) \neq v(\beta)$. Thus $\alpha \in Z_0$ and $\beta \notin Z_0$, or conversely. By (Z2), however, we have $Z_0 = Z(R)$, for some $R \in \underline{R}$. Denoting by R_0 one of the realizations for which that holds we get: $\alpha \in Z(R_0)$ and $\beta \notin Z(R_0)$, or conversely. Thus $R_0 \in M(\alpha)$ and $R_0 \notin M(\beta)$, or conversely, and so $M(\alpha) \neq M(\beta)$. Consequently (Z2) $\Rightarrow$ (3).

Conversely, assume $Z_0 \notin Z/\underline{R}/$. And take a map Z satisfying condition (Z1), but in such a way that $Z/\underline{R}/ = \{Z_1\}$, for some $Z_1 \in \underline{Z}$, with $Z_0 \neq Z_1$. (Obviously such a map always exists.) Then all realizations go over into just one complete set Z_1, and we get:

$$M(\alpha) = \begin{cases} \underline{R}, & \text{if } \alpha \in Z_1 \\ \emptyset, & \text{if } \alpha \notin Z_1. \end{cases}$$

Now take – as in Figure 1 – two propositions $\alpha, \beta \in Z_1$ such that $\alpha \in Z_0$ and $\beta \notin Z_0$. (Such two propositions always exist too, as we have always $Z_0 \cap Z_1 \neq \emptyset$, and as by maximality none of the two sets of propositions is included in the other.) Since both propositions belong to Z_1, we get $M(\alpha) = \underline{R} = M(\beta)$. And since α belongs to Z_0, and β does not, we get $v(\alpha) = 1$ and $v(\beta) = 0$, contradicting thesis (3). QED.

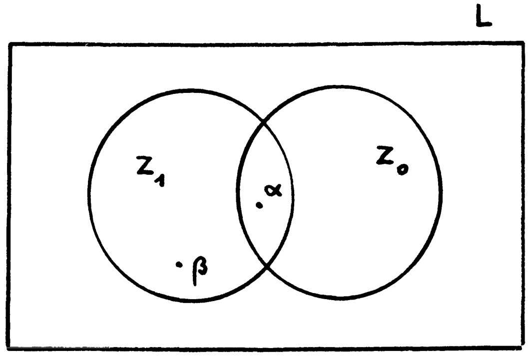

Fig. 1.

4. TRUTHS A PRIORI

The collection $\underline{Z}$ consists of all sets of propositions maximally consistent. Out of that collection the function Z selects a subcollection $Z/\underline{R}/$ of those sets which are "admissible" in view of the semantic system in question. Each member-set of $Z/\underline{R}/$ is so to say eligible to be the totality of truths.

The member-sets of $Z/\underline{R}/$ are theories, i.e. sets closed under the operation Cn. Consequently, their intersection $T_z = \cap Z/\underline{R}/$ is a theory too. And since $Z/\underline{R}/ \subset \underline{Z}$, we have $\cap\underline{Z} \subset \cap Z/\underline{R}/$. Thus the set of all tautologies $T_0 = \cap\underline{Z}$ is included in the set T_z, and we get the arrangement shown in Figure 2.

The members of the set T_z are propositions satisfied in all possible worlds. Thus we recognize in Figure 2 Kant's tripartite partition of judgements, as shown in Figure 3.

The set T_0 of all logical truths is determined by the abstract logic (L, Cn) alone, i.e. by the logical syntax of the language L. The set T_z of all propositions true *a priori*, however, depends also on the semantic tie Z connecting that abstract logic with reality. (But observe that T_z does not determine Z uniquely: for two different functions Z and Z', there may be well $T_z = T_{z'}$.)

Using the terminology of Carnap one might say that the members of the set T_z–T_0 – i.e. Kant's "synthetic propositions *a priori*" – are the "meaning postulates" of the language L: propositions true merely by force of their being some semantic tie between that language and reality.

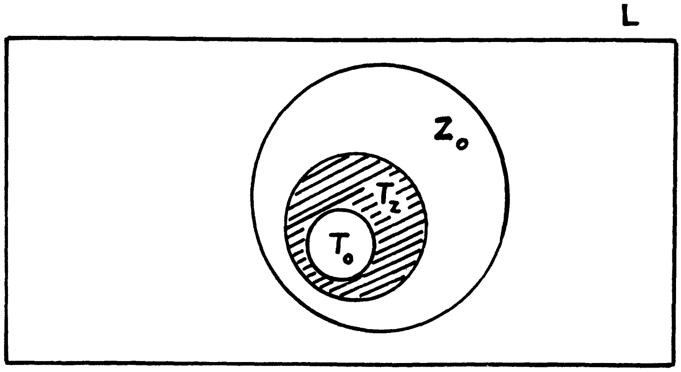

Fig. 2.

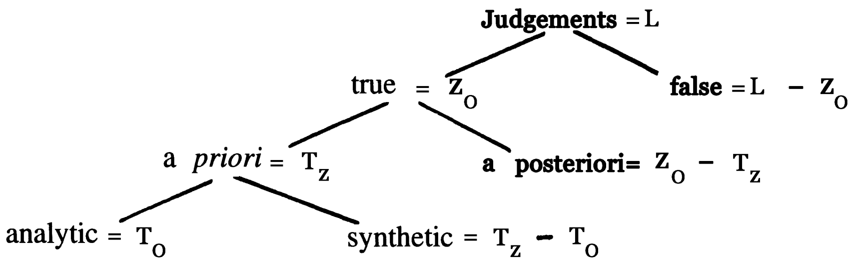

Fig. 3.

For radical empiricism, such as Hume's, the set mentioned is empty: there are no synthetic propositions true *a priori*, ie.. $T_z = T_0$. The *a priori* coincides then with the analytic, provided we take "analyticity" – like Wittgenstein, thesis 6.11 – in the wider Fregean sense to cover everything which is true alone by force of definition *and* of the laws of logic. (Observe that a special case of Humean empiricism results if the map Z is onto $\underline{Z}$, i.e. when $Z/\underline{R}/ = \underline{Z}$. Possibility coincides then with consistency.)

5. SYNTHETIC ENTAILMENT

The map Z is the way language is tied to reality by means of a corresponding logical space. In view of that tie another consequence operation may be defined on L. We denote it by Cn_z and call it "syn-

thetic consequence", defining it simply as follows: for every $\beta \in L$ and $A \subset L$,

$$\beta \in Cn_z A \Leftrightarrow \bigwedge_{Z \in Z/\underline{R}/} (A \subset Z \Rightarrow \beta \in Z).$$

It is easily checked that the operation Cn_z is extensive (i.e. $A \subset Cn_z A$) and monotonic (i.e. $A \subset B \Rightarrow Cn_z A \subset Cn_z B$). So let us see merely that it is idempotent too (i.e. $Cn_z Cn_z A \subset Cn_z A$).

To begin with, observe

(4) $\bigwedge_{Z \in Z/\underline{R}/} Cn_z Z \subset Z.$

For take any $\alpha \in Cn_z Z$. Then: $Z' \in Z/\underline{R}/ \Rightarrow (Z \subset Z' \Rightarrow \alpha \in Z')$. Substituting here Z for Z', we see that $\alpha \in Z$. QED.

Consequently,

(5) $\bigwedge_{Z \in Z/\underline{R}/} (A \subset Z \Rightarrow Cn_z A \subset Z).$

For if $A \subset Z$, then $Cn_z A \subset Cn_z Z$, by monotonicity. But $Z \in Z/\underline{R}/$, so $Cn_z A \subset Z$, in view of (4). QED.

Hence

(6) $Cn_z Cn_z A \subset Cn_z A.$

For suppose $\beta \in Cn_z Cn_z A$. This means that $Cn_z A \subset Z \Rightarrow \beta \in Z$, for every $Z \in Z/\underline{R}/$. But $Cn_z A \subset Z \Leftrightarrow A \subset Z$, by implication (5), the converse implication being obvious. Consequently, $A \subset Z \Rightarrow \beta \in Z$, for every $Z \in Z/\underline{R}/$. I.e., $\beta \in Cn_z A$. QED.

We also have the equality

(7) $T_z = Cn_z \emptyset.$

For $\beta \in Cn_z \emptyset$ iff $\emptyset \subset Z(R) \Rightarrow \beta \in Z(R)$, i.e. iff $\beta \in Z(R)$, for every $R \in \underline{R}$. And the latter means that $\beta \in T_z$. QED.

Setting $\alpha \Rightarrow \beta$ iff $\neg\alpha \vee \beta$, we see that the deduction theorem holds for Cn_z:

(8) $\beta \in Cn_z(A \cup \{\alpha\})$ iff $(\alpha \Rightarrow \beta) \in Cn_z A.$

Indeed, $\beta \in Cn_z(A \cup \{\alpha\})$ iff

$$\begin{array}{lll}
\text{iff} & \bigwedge_{R \in \underline{R}} & ((A \cup \{\alpha\}) \subset Z(R) \Rightarrow \beta \in Z(R)) \\
\text{iff} & & ((A \subset Z(R) \text{ and } \alpha \in Z(R)) \Rightarrow \beta \in Z(R)) \\
\text{iff} & & (A \subset Z(R) \Rightarrow (\alpha \in Z(R) \Rightarrow \beta \in Z(R))) \\
\text{iff} & & (A \subset Z(R) \Rightarrow (\alpha \notin Z(R) \text{ or } \beta \in Z(R))) \\
\text{iff} & & (A \subset Z(R) \Rightarrow (\neg\alpha \in Z(R) \text{ or } \beta \in Z(R))) \\
\text{iff} & & (A \subset Z(R) \Rightarrow (\neg\alpha \vee \beta) \in Z(R)),
\end{array}$$

with the first equivalence holding by definition, the second by the completeness of the set $Z(R)$, and the sixth by the fact that $Z(R)$ is a prime theory under Cn. QED.

As a corollary of (7) and (8) we get the equivalence:

(9) $(\alpha \Rightarrow \beta) \in T_z$ iff $\beta \in Cn_Z\{\alpha\}.$

Now by simple algebra of sets we see that the consequence Cn_z is regular, i.e. that we have

(10) $Cn_z A = \bigcap\{Z \in Z/\underline{R}/ : A \subset Z\}.$

Therefore, as $Z/\underline{R}/ \subset \underline{Z}$, we have also, for every $A \subset L$:

(11) $CnA \subset Cn_z A.$

Moreover, we have the equality $Cn_z(A \cup Cn_z B) = Cn_z(A \cup B)$. (Indeed, $A \subset Cn_z(A \cup B)$ and $Cn_z B \subset Cn_z(A \cup B)$; thus $A \cup Cn_z B \subset Cn_z(A \cup B)$, and so on). So $Cn_z(A \cup T_z) = Cn_z A$, which in view of (11) yields the inclusion

(12) $Cn(A \cup T_z) \subset Cn_z A.$

The converse of (12) fails, however, as not all the theories containing T_z need belong to the collection $Z/\underline{R}/$. This may be seen from the simple example shown in Figure 4. Suppose $Z/\underline{R}/ = \{Z_1, Z_2\}$. Then

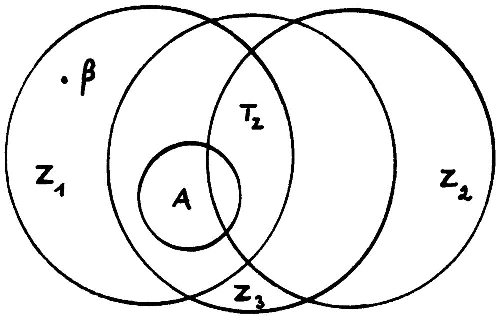

Fig. 4.

$T_z = Z_1 \cap Z_2$, and $\beta \in Cn_z A$. But $\beta \notin Cn(A \cup T_z)$, as the proposition β is separated from the set A by the set Z_3.

We have, however, the following theorem:

For every $A \subset L$,

$$(13) \quad Cn_z A \subset Cn(A \cup T_z) \text{ iff } \bigwedge_{\beta \in Cn_z A} \bigvee_{A_i \in \mathrm{Fin} A} \beta \in Cn_z A_i,$$

with Fin A being the collection of all the finite subsets of A.

Indeed, to begin with assume the left-hand part, and take any $\beta \in Cn_z A$. Then $\beta \in Cn(A \cup T_z)$ by assumption. And since the consequence Cn is finite, $\beta \in CnX$ for some $X \in \mathrm{Fin}(A \cup T_z)$. Now setting $A_i = A \cap X$ we have clearly: $A_i \in \mathrm{Fin}A$. Moreover, $X \subset A_i \cup T_z$, so $CnX \subset Cn(A_i \cup T_z)$. Hence $\beta \in Cn(A_i \cup T_z)$, which by (12) yields the conclusion: $\beta \in Cn_z A_i$.

Next assume the right-hand part, and again take any $\beta \in Cn_z A$. By assumption there is then an $A_i \in \mathrm{Fin}A$ such that $\beta \in Cn_z A_i$. Let α_i be the conjunction of all the members of A_i. By the definition of Cn_z we see that $CNn_z\{\alpha_i\} = Cn_z A_i$. So $\beta \in Cn_z\{\alpha_i\}$, which means by (9) that $(\alpha_i \Rightarrow \beta) \in T_z$. Hence by *modus ponens* we get: $\beta \in Cn(T_z \cup \{\alpha_i\})$.

Evidently, we have also $Cn\{\alpha_i\} = CnA_i$. Thus $Cn(T_z \cup Cn\{\alpha_i\}) = Cn(T_z \cup CnA_i)$, and so $Cn(T_z \cup \{\alpha_i\}) = Cn(T_z \cup A_i)$. In view of the foregoing result the latter identity yields: $\beta \in Cn(A_i) \cup T_z)$. So $\beta \in Cn(A \cup T_z)$, as $A_i \subset A$. QED.

6. A FINAL REMARK

The set T_z encompasses all of mathematics. Thus on our assumptions mathematics neither reflects reality, nor is it part of the logical syntax of language. Now, mathematics is a manifestation of the way language and reality are tied together. Logic does not depend on that way, but mathematics does – unless we take the stand of a Russellian logicism expressed by the equation $T_z = T_0$, and saying that all mathematics is just an expansion of logic.

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REALISM VS RELATIVISM IN PHILOSOPHY OF SCIENCE (SOME COMMENTS ON TARSKI'S THEORY OF TRUTH)*

0. INTRODUCTION

Realism need not to be opposed to relativism. It may certainly be opposed to various other positions.[1] But, as I maintain in this paper, in philosophy of science the two doctrines play complementary roles. Consequently, every departure from realism, the philosophy which I defend, is a step towards accommodating some views characteristic of relativism. At the same time, it is obvious that uncompromising, orthodox realism is untenable. In particular, I argue that any acceptable version of realism must substantially revise one of the main components of the doctrine – Tarski's conception of truth.

1. TWO PHILOSOPHIES OF PHILOSOPHY OF SCIENCE

What are scientific theories suppose to tell us about the world we live in? Are they meant to provide an adequate description of the world as it really is, or of the world as we are able to perceive it, i.e. as it is accessible to us through sensory experience? Reality 'as we see it' need not be the same as reality 'as it actually is'. Or perhaps we should settle for still less. There may be little doubt that the process of acquiring knowledge cannot be carried out without some preconceived, *a priori* ideas. Thus one may argue that any knowledge we are able to get must be to some extent arbitrary; we are not able to describe the world as it is or even as we perceive it. The world we describe is the world which is created by us, molded from the real one so as to fit our conceptual apparatus, the knowledge we already have, our beliefs and even what we expect to be true.

The view that the knowledge accumulated in scientific theories refers to reality as it actually is presupposes that to whatever aspect and/or part of the real world a scientific theory may refer, it has a certain structure determined by the objects it involves and the variables (prop-

Jan Woleński (ed.), Philosophical Logic in Poland, 337–361.

erties and relations) in terms of which these objects can be described. It also presupposes that we are able fairly adequately, though perhaps incompletely, to learn this structure. Moreover we are able to do this by forming a conceptual apparatus that matches the structure and thus allows us both to examine and to account for the regularities characteristic of it. The three presuppositions just discussed are the central assumptions of *scientific realism.*[2]

The thesis that the process of learning about reality is actually the process of constructing it need not be taken literally. The main claim expressed by this thesis is the following: there is no unique true or adequate account of the world; the world has no definite structure. Rather it is a sort of cluster of various 'spots' from which, by the faculty of our imagination, we form various *Gestalten.* It is largely up to us whether we see a 'rabbit' or a 'duck' – in fact there is neither rabbit nor duck. We merely describe what we see by saying that we see one or another, for this allows us to provide a condensed and suggestive account of our experience. Whatever we settle for, our decision has further consequences.

Once a process of viewing reality one way or another is started, we tend to stick to it. We improve it, extend it by adding new details – if originally, we saw just a rabbit, now we see that the rabbit has floppy ears and hides between two shrubs. At the same time we become less and less able to see anything which does not fit the picture we discovered – in fact, invented. In this sense reality is a product of our imagination and, if our ability to make a critical appraisal of the process of knowledge acquiring fails, we start believing that the description of what we see is a description of what is there, is a description of real things, real states of affairs and real events. Actually, any truth which is available to us is related to one or another *epistemic perspective* we are able to form, i.e. to one or another way of viewing things, evaluating empirical data, judging views relevant to ours. This philosophical doctrine is known as *(scientific) relativism.*

Scientific relativism underlies, if it is not a part of, two quite different trends in philosophy of science. One of them is conventionalism – the doctrine put forward by H. Poincaré, P. Duhem, E. Le Roy and, among others, followed by K. Ajdukiewicz, R. Carnap (at least in some periods of their philosophical activity), O. Neurath and, more recently, in a restricted form by W. V. Quine. The other is the doctrine of the incommensurability of successive scientific theories; two theories are

incommensurable if the conceptual apparatuses they use are so radically different that the claims of any one of them cannot be adequately translated into the other.[3]. The doctrine is usually credited to T. Kuhn whose idea of science developing through revolutionary changes is closely related to that of incommensurability.[4] But Kuhn has at least two predecessors.

One of them is L. Fleck, who stated (in 1935) the basic assumptions of the doctrine, even though he did not use the term 'incommensurability'. In Fleck's terminology a switch from one theory to another incommensurable with the former is a switch from one 'style of thought', *Denkstill*, to another.

The second, though in a much less obvious way, was Ajdukiewicz during the period of his *radical conventionalism* (1934). The conventionalist need not maintain that the evolution of science must result in incommensurable theories, but should be ready to admit that alternative theories may happen to be incommensurable. One of the central claims of Ajdukiewicz's conventionalism (expressed without using the term 'incommensurability') was that this, in fact, must happen.[5]

The history of the doctrine of incommensurability cannot be discussed here unless we go far beyond the topic of our main interest. But one thing seems to be worth noticing. The philosophies of conventionalism and of incommensurability have entirely different origin. Let us dwell on this point for a while.

Conventionalism is a doctrine derived from a purely logical analysis of science and its dynamics. On this view, alternative theories providing alternative pictures (commensurable or not) of the same part or aspect of reality are alternative solutions to the same logical problem. This is to find a theory which, on the one hand, meets some formal criteria (notably: consistency, elegancy, simplicity) and some practical expectations, and on the other hand, fits the available empirical data. Since the problem may not have a unique solution, the growth of science may not be uniquely determined; science may admit various theories of the same phenomenon, none of them better than the others.

Now, the incommensurability thesis is primarily motivated by sociological considerations. This is especially obvious in the case of Fleck, whose interest in logical factors in the growth of science was next to none; but it is also visible in Kuhn's approach. The central thesis of Fleck's conception of science is that science is a collective enterprise. Scientific ideas, conceptions, theories result from interaction among the

members of a group of people who strive for unification of their conceptual apparatus, their views, and their methodology. It is irrelevant to the present discussion how this goal is achieved: by some participants accepting the authority of other members of the group, by mutual discussion, or perhaps by a combination of these two methods. Once it is achieved, the group becomes a *Denkkollektiv*. The members of the group share views essential to their scientific work, interpret empirical evidence available to them in the same way, are ready to accept the same solutions to the difficulties their views may encounter, etc., etc. If X belongs to the 'rabbit' and Y to the 'duck' *Denkkollektiv*, X and Y cannot understand each other, they think and act in entirely different ways.

Fleck rejects the idea that of two styles of thought one may be better than the other in some absolute sense. Each *Denkstill* may be good for certain purposes and bad for others; none may be evaluated outside the context in which it has been formed. The magical way of thinking of primitive people serves their needs; similarly the 'scientific' way of thinking of educated people at the end of the twentieth century serves theirs. Neither is superior to the other. Neither is false; moreover, there is no single claim which can be declared true or false outside of a specific *Denkstill*.

In his relativism Fleck is radical and uncompromising. He does not hesitate to speak about the relativity of reality and the relativity of truth and to treat these claims literally. Some relativists do. Some of them prefer to interpret the thesis about the construction of reality as a metaphor rather than a description of what actually takes place when we switch from one epistemic perspective to another. Also, they prefer not to apply the notion of truth to those views which depend on a specific epistemic perspective, restricting its use to those statements which they believe to be independent of any epistemic perspective whatsoever (Fleck would deny the very existence of such statements). In the accepted terminology they are *instrumentalists* rather than relativists. This, in particular, applies to Kuhn who resists being classified as a relativist.[6]

The two views, realism and relativism, described above may take different forms. Rendered radically, they are dramatically divergent and cannot be combined in a consistent whole. Construed moderately, they are just complementary. Consequently, the way in which these two doctrines are related one to another cannot be adequately accounted for

by saying that they are opposite views. No realist can deny that the way in which we perceive things surrounding us is selective, depends on our conceptual framework, our knowledge, our beliefs and even superstitions. But the realist hopes that, regardless of how difficult it may be, we are able to distinguish things from their appearances. The relativist does not.

In its orthodox form neither view is likely to be defensible. No wonder then, that those philosophers of science who identify themselves with the realist position incline towards one or another version of *pragmatical realism* to use the expression coined by H. Putnam,[7] i.e., realism which does not ignore the fact that whatever knowledge we may have is conditioned by our human limitations. For a realist ready critically to appraise his or her views, relativism, even in its most radical form put forward by L. Fleck, is by no means an odd idea which may be ignored. Quite to the contrary, it is an intellectual challenge which must be taken seriously.

2. FACTICITY REALISM

This is the right place to go back and comment upon the tentative definition of scientific realism given earlier.

Most importantly, notice that scientific realism is more than the metaphysical tenet that there is an outer world. This belief underlies scientific realism, but it is taken for granted rather than argued. The main concern of scientific realism is the accessibility of adequate knowledge about reality, scientific knowledge in the first place, and the main claim is that such knowledge is accessible. The question arises what is to be meant by adequate knowledge. Different answers to it define different variants of the realist doctrine.

Adequate knowledge should be both sufficiently complete and sufficiently accurate. Consequently, the boldest and the most optimistic variant of scientific realism is one which claims that whatever regularities are displayed by the phenomena surrounding us, science is able to discover and then describe them in a fully accurate way. I definitely prefer to start my discussion with something more modest, not to say less naive.

Bas C. van Fraassen (1980) characterizes scientific realism as the view that 'scientific theory construction aims to give us a literally true story of what the world is like and that acceptance of a scientific theory

involves the believe that it is true'. Nancy Cartwright (1983), who defines scientific realism in a similar way, calls the view that laws of nature describe facts about reality the *facticity view*. I will use this term and refer to scientific realism in the sense defined by van Fraassen as *facticity realism*.

The main advantage of van Fraassen's definition of realism is that it grasps in a few unambiguous words the essence of the realist doctrine. To be sure, van Fraassen is not telling us that to be a realist is to believe that scientific theories are literally true. He merely tells us that to be a realist is to believe that literally true theories of empirical phenomena are what scientists seek to achieve. The main disadvantage of this definition is that all its straightforward interpretations render scientific realism a lost cause.

It is a fact of life that, by and large, scientific theories are not literally true. This does not imply that no literally true theories are available, but there are reasons to doubt whether 'a literally true story of what the world is like' will ever be given. The issue should perhaps be discussed in some detail, and I am going to undertake it in the next section, but certainly most scientists and most philosophers of science would agree that if literal truth is available, it is available on a 'local' rather than a 'global' scale. There are many scientific claims that practically all experts consider to be established beyond any doubt but, as a rule, they are claims which refer to specific, singular states of affairs. The more general a scientific claim, the broader the scope of a scientific theory, the less likely it is that it will turn out to be accurate in every respect. At least that is the lesson we have been able to learn from the history of science.

One way to react to these observations is to say that the requirement of literal truth should not be treated literally. Rather it sets an ideal which, although it cannot be achieved, should be pursued. Accordingly, to be a facticity realist is to believe that science, through successive transformations of its theories, brings us closer and closer to truth, though perhaps no fully accurate account of world's regularities will ever be achieved within any finite period of time. The idea of truth meant as the limit of a potentially endless series of transformations of human cognition has a long history and a long list of proponents. Not all of them have been motivated by their interest in philosophy of science. But among those who have are such outstanding philosophers as Charles S. Peirce and, more recently, Karl Popper.[8]

An obvious question to be asked is how one is able to measure the distance from truth or, to put this question in qualitative rather than quantitative terms, how one may know that of two theories one is closer to truth than another. There are two ways in which the question has been approached. One is Popper's idea of verisimilitude (1983), the other is the conception of approximate truth. Several variants of the latter have been put forward; one, which consists in accommodating in an appropriate manner Tarski's conception of truth, is to be found in Wójcicki (1979).[9]. Although the ideas of verisimilitude and approximate truth are different, they are plagued by the same difficulty. As D. W. Miller (1974), R. Tichy (1974) and J. H. Harris (1974) show in their criticism of Popper's conception of verisimilitude, if of two competing theories both imply some false statements, then no purely logical criteria allow us to conclude that one of them is closer to the truth than the other.

Neither the radical version of the facticity view (truth is achievable) nor its relaxed version (truth can be approached) is defensible. Or rather, I do not believe that either is. Rather than go through the agony of a hopeless search for logical criteria of verisimilitude, approximate truth or any other form the idea of closeness to truth may take, we should reexamine the very notion of truth.

3. SOME COMMENTS ON TARSKI'S CONCEPTION OF TRUTH

The official theory of truth of the doctrine of scientific realism is Tarski's conception of truth.[10] This conception is often referred to as the *correspondence theory* of truth, although the adequacy of this classification may be questioned (see S. Haack, 1978 and 1987). Three aspects of Tarski's theory should be carefully separated.

Given a specific state of affairs, we want to provide an adequate account of it. In order to do so, an appropriate conceptual apparatus should be formed; the terms in which the account is to be given should be selected and assigned appropriate referents. This aspect of the theory of truth will be referred to as *epistemological.*

Now suppose that both the state of affairs and a language in which a true description of it can be provided are given. Then, the obvious question to be asked is how the users of the language may know that a specific claim is true. The aspects of the theory of truth which concerns the criteria of truth will be referred to as *methodological.*

Finally there is a *logical* aspect of the theory, which concerns the question of how and under which conditions the concept of truth for a given language, related to a given state of affairs (a given 'semantic model' in the logical terminology[11]), can be defined.

Tarski's primary interest was the logical aspect of the theory of truth. The main result of his study consists in showing that if both the language and the corresponding model satisfy certain formal conditions (most importantly: the language should meet the conditions imposed by Tarski on *formalized object languages*, and the set theoretic model should be given in the form of a set-theoretic structure whose *similarity type* corresponds to the *signature* of the language), then in a suitably selected *metalanguage* a rigorous definition of truth for the language in question can be given.

As was stressed by Tarski, the definition of truth need not yield any criterion of truth. In fact, if the Tarski approach is scrupulously followed, it does not. This observations, however, needs a qualification, for Tarski's definition of truth does yield some *relative* criteria. An immediate example of such a criterion is the clause of the definition that says that the disjunction A or B is true if and only if at least one of the disjuncts is true.

The only sentences for which Tarski's definition of truth apparently does not provide any criterion of truth, either absolute or relative, are *atomic* sentences, i.e. the simplest sentences of the language of which all the remaining ones are built. In their case the theory offers us very little indeed, because for each such sentence A, the definition directly reduces to Tarski's convention T:

A is true if and only if $\underline{A}$,

where $\underline{A}$ is the translation of A in the metalanguage.

In fact, the case of the atomic sentences also needs some qualification, for in general one may expect that the translation $\underline{A}$ may offer some cues how to decide the sentence A. This possibility however is ruled by Tarski's requirement that the object language should coincide with the object part of the metalanguage. Consequently, translations take entirely idle form, illustrated by Tarski's celebrated example:

snow is white is true if and only if snow is white.

Some authors argue that Tarski's definition of truth tells us in a suitably selected metalanguage which sentences of the object language are true, but it does not tell us why we should consider them as true, thus

it does not provide any philosophical analysis of truth.[12] If one expects a definition to provide the meaning of the term defined rather than only to determine its extension, this criticism may be justified. Tarski's definition does not explicitly involve any philosophical analysis of the notion of truth. On the other hand it certainly is based on such an analysis. Tarski presupposes that the truth value of atomic sentences is determined by referents of the terms of which those sentences are built and the way in which the definition is constructed makes it clear how the latter determine the former. The truth value of compound sentences is assumed to be determined by the truth values of the atomic ones and the meaning of the logical constants by which the former are built of the latter. Accordingly, the remaining clauses of the definition explicate the meaning of logical connectives and quantifiers.

The next remark which is worth making is the following. Tarski's conception of truth presupposes that truth is a property of sentences of a language given in advance. But, of course, one who believes that the truth is what was written in the Book of Nature does not necessarily believe that this book was written in Sanskrit or Hebrew. The simplest (and, unfortunately, also simplified) answer to this, rather artificial, problem is the following. Scientists may use different languages; they may use different native languages, different symbolic notations and even different mathematical apparatuses. But as long as they know how to adequately translate beliefs stated in one of these languages into another, which language is used does not matter. So, we may treat all such languages as one language, and we may treat the rules of translation they use as logical rules of derivation. In this way, the definition stated for any of these languages extends to all the remaining ones. Of course, the idea of the super-language is merely a roundabout way to explain how one who uses the notion of truth, meant to be a linguistic notion, may still not feel to be committed to use any particular language.

The last remark I want to make is directly relevant to the central topic of this paper. According to some relativists, the main reason of inadequacy of Tarski's theory of truth is its "bipolarity" – the fact that it forces us to treat every sentence as having a determinate truth value, thus as being either true or false. On the logical ground bipolarity of Tarski's conception was questioned by M. Dummett.[13] Among philosophers, J. Margolis considers the stand on the bipolarity issue to be a criterion for indicating the position that one takes in the debate between realists and relativists.[14]

Tarski's theory of truth is bipolar indeed, but it can be easily extended to the languages with truth-value gaps, i.e., the languages in which some sentences have no truth value like, e.g., all expressions of the form $x : 0$ in arithmetic. Also, there is nothing in the doctrine of realism which bans using such languages. On the other hand, hardly any relativist, certainly neither Dummett nor Margolis advocate many-valued logic. I incline to believe, therefore, that the differences between realists and relativists concerning the notion of truth have little to do with the problem of bipolarity, and the problem itself was misconceived.

4. DOES TARSKI'S THEORY OF TRUTH APPLY TO THE LANGUAGE OF SCIENCE?

Since the doctrine of scientific realism is the doctrine of accessibility of truth in the sense which Tarski was trying to grasp, for the realist a negative answer to this question most pose a serious problem. Now, although Tarski's conception works perfectly in mathematics, it is by no means obvious that it extends in any natural way to the whole language of science. I am going to discuss this matter in some detail.

Not all the phenomena surrounding us are clearly determined. In fact, none are. All are more or less 'vague', 'fuzzy', 'blurred', 'nebulous', 'shady', to mention words most commonly used to describe this. Of course, there are extremes on this scale. We can fairly easily identify one person as taller than another, but we may find it very difficult to decide who is healthier than whom. The complex nature of the phenomenon of health is an explanation, but, first, we cannot help dealing with complex phenomena and, second, even relatively simple phenomena may pose problems of identification. But, regardless of how vague a phenomenon may be, if it is of importance or of interest for us, we need a name for it – a word which stands for it. We need a name in order to keep records about the phenomenon and to communicate our beliefs about it. Also, by giving a name to a phenomenon we call attention to it, sometimes successfully. Introducing a new name may result in a significant shift of interest in the relevant discipline; witness 'Gestalt' in psychology, 'tropism' in biology, 'anti-particle' in physics, 'stress' in medicine, 'paradigm' in philosophy of science.

The fact that words refer to blurred phenomena makes the relation of reference *unstable* – vulnerable to various modifications. A blurred phenomenon may be 'shaped' one way or another; trying to make it

more determinate we tend to examine different possibilities of drawing its border lines in a more precise way. Now, while the referent undergoes various modifications, the word is usually kept unchanged. This is what I have in mind speaking about the instability of the reference relation.

It may be worth dwelling for a while on the vagaries of reference. Modifications of the reference relation are often *ad hoc*; the process of 'shaping' phenomena is to some extent random. But it may as well be systematic, based on some methodological principles. The latter is characteristic of science, where it is commonly agreed that in order for a phenomenon to be an object of scientific inquiry, it must satisfy certain conditions. One of them, of special significance, is the condition of repeatability. In order for an alleged phenomenon to count as a real one, not mere an artifact or a product of sheer imagination, it must be reproducible at various time, in various places and in various circumstances. To reproduce a phenomenon means, obviously, to produce another instance of it; the term phenomenon may mean either *phenomenon-type* or *phenomenon-instance*, i.e. a specific, concrete, single case of a phenomenon type.

Reproducibility is one of a number of general methodological requirement that scientists expect phenomena they study to satisfy. Thus, for instance, they focus their attentions primarily on those phenomena which are as tightly related to others as possible. Incidentally, any relation between phenomena is also a phenomenon.[15] The requirement of strict correlation between phenomena renders the reference relation (or, more broadly, meaning) holistic;[16] the process of assigning referents is not a piecemeal process, decisions concerning the referent of one word depending on and affecting decisions concerning the others.

In addition to general methodological principles, there are special expectations characteristic of the various given disciplines. Not all of them take the form of explicit, clearly stated rules. But all contribute to the instability of the reference relation, inviting if not forcing its successive genuine or merely apparent improvements. Needless to say, various methodological requirement may well be inconsistent, and thus the adaptation of the referent to the methodological requirements cannot be achieved without some trading off.

This fairly complex picture becomes still more complicated, if we take into account that, typically, language does not just serve to communicate that such and such state of affairs obtains, but it serves to communicate this for a certain specific purpose. Consequently, speaking e.g. about

the health of an individual or group in one context, we may have in mind quite a different aspect of this phenomenon than we do have when we are speaking about health in another context. This results in *conceptual relativity* – a semantic phenomenon on which some theorists, notably F. I. Dretske (1981) and H. Putnam (1989), have recently focused their attention. Note that conceptual relativity is not the same as 'direct' ambiguity which consists in the same word serving as a name for entirely different objects: e.g. 'box' as a case and 'box' as a blow. The ambiguity caused by conceptual relativity is more subtle. The same word can have different referents in different contexts, but all the referents have something in common, as e.g. the word 'empty' in 'empty pocket' and 'empty house' or 'small' in 'small house' and 'small elephant'.[17]

Often we know very well how to use the same word to speak about different things and make ourselves perfectly intelligible. On other occasions we struggle with semantic difficulties seeking the best way to express ourselves. It is obvious that a language in which every shift of reference and any shade of meaning is represented by a different word would be monstrously huge. But even if we were able to learn it, it would most likely lack much of the semantic power of the ordinary language. It would be lacking *referential flexibility*, the property that allows us, without breaking the rules of a language, to intelligibly use the same word to speak about different things, and thus not only to express ourselves more economically but often, by exploiting associations thus induced, more adequately.

Tarski's definition of truth deliberately ignores all vagaries of the reference relation. An excuse sometimes offered is that Tarski was concerned with scientific rather than ordinary language. But, as I have been arguing, none of the phenomena discussed above, vagueness of referent, instability of the reference relation, conceptual relativity and holism of meaning is restricted to ordinary discourse. They are present and, moreover, present in a most substantial way, in the language of science, the language of the fully formalized part mathematics being, I believe, the only exception.

Certainly I am not the first to notice that Tarski's conception does not apply in any direct way to non-mathematical theories. But some people hold the view that the referential flexibility of scientific discourse is characteristic of those stages in the formation of science which precede final and precise formulation of theory, and thus it should be treated as a symptom of immaturity of the relevant part of science. This, I am

afraid, is an illusion. The section which follows is entirely devoted to this issue.

5. INTERPRETING PHYSICAL THEORIES

While in the previous section I was concerned with the language of science as a whole, the present analysis will be confined to Newtonian Mechanics, a theory which on the one hand was superseded by relativistic mechanics and thus is a closed part of physics and, on the other, is still very much in use. Although many other parts of physics could serve my purpose, the choice is not accidental. The formal structure of NM is relatively simple and the theory belongs to those with which practically everybody has some familiarity. Rather than try to argue in a direct way that the language of NM displays all the characteristics features of flexibility I have discussed in the previous section, I am going to do this indirectly. First, I will single out and describe some of the difficulties one encounters when trying to construct a Tarski-style semantics for this theory. When this is done, I am going to argue that these difficulties are due to the fact that in order to render the laws of NM in an adequate way, the language in which they are construed must be flexible, in just the sense discussed in the previous section.

Consider Newton's First Law,

$$(*) \qquad \mathbf{F} = mD^2\mathbf{x}.$$

It relates three variables (quantities): force, mass and position. The first and the last are vectors, mass is scalar. D^2 is the mathematical operation of forming the second derivative over time; both $\mathbf{x}$ and $\mathbf{F}$ depend on time (are time-indexed variables). Thus, in fact, time is one more variable which the law involves. From the mathematical standpoint the law is a general schema of identities which one obtains by substituting the position, the mass and the force by relevant mathematical functions defined within an abstract mathematical setting, which consists of three-dimensional Euclidean space, the one-dimensional continuum of real numbers (referred to as 'time'), and a frame of reference.

At the very moment one leaves mathematics and starts thinking about $\mathbf{F} = mD^2\mathbf{x}$ and other laws of NM as a piece of physics, all the components of the mathematical setting, as well as all mathematical variables the laws involve must be replaced by their *intended* physical

counterparts. Jointly those physical counterparts will form the *intended* semantic model for the language of Newtonian Mechanics, i.e. the one the theory is supposed to describe. This replacement, referred to as *physical interpretation*, is by no means a straightforward procedure. Still worse, there are reasons to doubt whether it can be executed in any acceptable way.

One, rather substantial reason, as is well known today, is that physical space is not Euclidean, and moreover time and space are not independent. This means either that the intended interpretation ignores reality or that the theory on the whole is false with respect to it. Apparently, of the two choices only the latter is acceptable, for the idea of examining the adequacy of laws with respect to a model which does not represent the actual state of affairs seems to be pointless. And it is pointless, unless one believes that although any semantic model with Euclidean space and independent absolute time does not represent reality faithfully, one may still learn something about how the Newtonian theory refers to the real world by examining it.

Actually, some theorists go even further and argue that, after all, empirical theories do not refer to the real world in any "direct" way. They refer to some idealized versions of it. Thus, for instance, the Euclidean space and absolute time are idealized counterparts of the real world and they are just those counterparts in which Newton located the movement of material bodies. What we have to do is to examine the adequacy of Newtonian Mechanics with respect to its 'natural' ontological setting, thus with respect to the idealized world with absolute time and absolute space. We should not expect empirical theories to be true, but rather require them to be *counterfactually* true, i.e. they would be true if the world were as its idealization. In this paper, I have no time to discuss this position advocated by L. Nowak (1980), and recently endorsed by N. Cartwright (1989), but I am skeptical about its adequacy.

The 'counterfactual option' being – rightly or wrongly – discarded, we are left with a seemingly unavoidable conclusion that the laws of Newtonian Mechanics are simply false. Note that this conclusion has been reached without any reference to Tarski's theory of truth. The whole argument was based on what we know about the real world from theories that we consider more reliable than Newtonian Mechanics. Although in general, this is an excellent way of arguing, in the context of the present discussion, it might be good to know how this conclusion can be justified by appealing to Tarski's semantic analyses.

Suppose, then, we ignore the inconsistency of NM with contemporary physics, and look for some 'independent' judgement about the truth-value of Newtonian laws, for instance that of his first law $\mathbf{F} = mD^2\mathbf{x}$, cited above as $(*)$.

To begin with note that the formula $(*)$ cannot be applied to any material body whatsoever, but – as one may learn from a relevant textbook – only those which can be treated as *mass points*. Furthermore, note that if the law in question, or in fact any law whatsoever is applied to a specific object a then it can be applied to that object only during its "life time". Thus, in fact $(*)$ should be treated as a shorthand for the sentence:

$(**)$ For every mass points a and every moment t in its life time $\mathbf{F}(a,t) = m(a,t)D^2\mathbf{x}(a,t)$.

In order to carry out on a full scale the Tarski-style analysis of the truth value of $(**)$ it would be necessary to state truth conditions for sentences which involve the two mathematical operators appearing in $(**)$, namely D^2 and the multiplication operator (the latter intervenes in $(**)$ even though it is not written explicitly). But there is no need to take such a heroic step: the relevant analysis of the two operators would be a rather involved task.[18] At this juncture my further effort will be directed towards two things: first, showing that the claim about the flexibility of the language of NM is correct and, second, showing that that flexibility makes the Tarski conception inapplicable to this theory.

The first part of this task is easy. To begin with, note that whether a given body can be treated as a mass point or not, thus whether or not the laws of NM can be applied to it, depends not only on the physical characteristics of both the body itself and its surrounding but also on the problem one wants to solve. Indeed, when NM is supposed to serve us to calculate the movement of planets in the solar system within a certain determined, 'not too long' interval of time and with some 'high but still not the best available' accuracy, all bodies in the system can be treated as mass points. Of course, in order to define the application in question in a fully accurate way, the expressions in the quotation marks should be fully specified.

Thus, as the example has been intended to show, in some applications the celestial bodies of which the planetary system consists can be treated as mass points. But, for instance, one way of explaining why the movement of Mercury does not satisfy Kepler's law consists in taking into account the oblateness of the Sun, a feature which no mass point

may have. In this application Mercury and the other planets are still treated as mass points, but the Sun is not. And, of course, if one tries to explain the phenomenon of the oceanic tides (another paradigmatic application of Newtonian Mechanics) treating the Earth as a mass point is just nonsense.

These rather straightforward observations prove immediately that the notion of mass point is a 'clinical' example of conceptual relativity. It is also vague (unless it is treated as a purely mathematical notion, but this is not the case I am discussing), for certainly there can be no clear-cut line between the cases when a body with respect to the problem examined is a mass point and when it is not.

Since the 'life time' of a body is actually the life time of the body treated as a mass point, both the vagueness and the conceptual relativity of the notion of a mass point are inherited by that of 'life time'.

Now, suppose that one uses Newton's laws in order to solve a question concerning the movement of a missile (one more paradigmatic example of application of NM). Depending on various physical factors, which would be a nuisance to list, and of course depending on the question asked, it may suffice to take into account the gravitational force and the friction caused by the forward movement of the missile. But it may not. It may be necessary to take into account specific weather conditions (wind, rain), to take into account the temperature and the spin movement of the missile, to calculate the effect of the Coriolis force, etc. Whatever may affect the movement of the missile is represented in the Newtonian analysis by a specific force. Of course, the existence of these forces does not depend on the question asked, but the nature of the question decides which of them can be ignored, and how accurately we should estimate those which cannot.

Let us have a bit more general picture of the situation. In order to apply Tarski's theory of truth to the laws of NM, one must start by assigning referents to all the physical terms in which these laws are stated. One must also define the sets of objects to which these laws are applicable, as well as defining the relevant spatio-temporal notions. As it follows from the above analysis, at least some of these referents depend one way or another on specific application of NM, thus in particular on the problem one expects to solve. That Tarski's semantics does not cover situations like this is obvious. But one may hope that each specific application can be represented by a separate semantic model in the standard Tarskian sense.

P. Suppes' multireferential semantics (1967), as well as its more sophisticated version developed by J. Sneed (1971) and other representatives of 'structuralist' philosophy of science[19] can be viewed as an implementation of this idea. I followed the multireferential approach in (1979) myself, also trying to cope with the problem resulting from the fact that mathematical formulation of empirical laws is stated in fully precise terms (is 'pointwise') while any physically meaningful interpretation of them is approximate. Now, however, I incline to believe that this approach misses an important point. Speaking in most general terms, I am afraid that each semantic model of a specific application of NM which is adequate to handle the question asked is inadequate for some other questions to which it apparently provides a solution. To be a bit more specific: I am afraid that the criticism that destroyed Popper's idea of verisimilitude,[20] extends to approximate truth.

What makes Tarski's theory of truth so attractive for those philosophers who share the realistic position, is that this theory is purely semantical, not 'contaminated' by any pragmatical considerations, and thus is not distorted by any relativist yearnings. But this alleged advantage is its main weakness. I will try to explain what I have in mind.

To know the laws of Newtonian Mechanics and to know where, when, and how these laws can be applied are two different things. And this practical problem underlies a conceptual problem. The way in which the concepts of NM are to be understood is certainly not defined by the laws of the discipline alone. Every new paradigm of a successful application of NM (e.g. the derivation of Boyle's Law of Gases with the help of the laws of NM, or using these laws to predict the movement of electrons in a magnetic field, or the movement of ionized molecules in the Millikan experiment) contributes to adequate understanding of concepts of NM and is a step towards determining the *scope* of the theory. Note that in none of these cases, could one know in advance whether calculations based on the laws of NM would agree with the empirical results or not. Nor could one in the case of any discovery which shows that these concepts are inadequate for some domain, that for instance, the turbulent flow of liquid cannot be adequately described with any reasonable accuracy with the aid of Newton's laws, or that the Newtonian concept of position is largely inadequate for dealing with quantum mechanical phenomena.

Somewhat metaphorically, the situation may be described as follows. Newtonian Mechanics is a set of ideas (part of which are its laws) which

jointly provide an attempt to describe the phenomenon of mechanical movement. Like all phenomena, this phenomenon is vague; neither its components nor its "boundaries" are clearly defined. All known paradigmatic applications of NM as well as all discoveries of the non-applicability of Newtonian laws contribute to better understanding of the phenomenon. Although a complete understanding is never achievable, still paradigmatic examples provide the users of NM with some general cues as to how to apply the theory correctly. No algorithm, no complete instruction defining the class of potential applications of NM and the way in which they have to be executed, can be given. This part of knowledge is practical knowledge, available by training.

To summarize: what would count as a possible application of the theory, and how specific applications are to be carried out escape any verbal definition. Only experts, relying on their competence, are able to distinguish proper applications from improper ones and can tell a correct implementation of an application from an incorrect one. If their judgements are wrong, then the theory turns out to be inadequate (or false) for some applications for which it is expected to work properly. Einstein's discovery of relativity proved the inadequacy of NM for all applications in which the phenomenon of relativity of space and time should be taken into account. But, although Newtonian Mechanics was eventually superseded by the Einsteinian Mechanics, EM, it was not rejected. The emergence of EM allowed for defining in a more accurate way the scope of applicability of NM and keeping NM as fully adequate with respect to it.[21]

Indeed, the experts know very well how to use the old theory, and on many occasions they find using it easier and more natural than its relativistic counterpart. This does not mean that they may not misjudge certain situations; certainly this may happen. What is worth noticing is they believe that it is impossible for NM to have successful applications in which EM does not work, which certainly means that EM is superior to NM at least in the described sense. Actually there are situations in which NM apparently works better than EM, but they are treated as an indication that the phenomenon studied is not properly understood.[22] The superiority of EM with respect to NM seems to be well established indeed.

This superiority, and thus the corresponding idea of progress in science as well, are defined by appealing to the competence of experts, thus in pragmatical terms.[23] Note that this approach has allowed us not

to become involved in the problem of whether Newtonian time and relativistic time, or Newtonian mass and relativistic mass, etc., have something in common. The meaning of these notions has certainly undergone considerable changes which are reflected both by the fact that EM is considered to be superior to NM and by the fact that it was necessary to redefine the scope of the latter. These changes of meaning do not prove, however, that the two theories refer to different things, thus e.g. besides the Newtonian and the Einsteinian concept of mass there are the Newtonian mass and the Einsteinian mass. I share the view of those people who incline to believe that Newton's theory and Einstein's theory both refer to the same mass, but the latter in a more adequate way. Of course, this belief has a typical realistic flavor.

The question to which this discussion gives rise is whether there is a notion of truth which allows us to grasp in a more precise manner the very vague idea of adequacy I have outlined above. I believe there is, but the topic must be discussed in a separate paper.

Any pragmatic idea of truth, pragmatic in the sense that it relates truth to human beliefs, attitudes and/or actions, is a departure from realism and a step towards relativism. But, as I keep saying, this step must be made. Even if Tarski's definition of truth were acceptable without the slightest modification, any attempt to use this definition in order to decide the truth value of a specific sentence would immediately force us to cope with pragmatical problems.

Whether we accept Tarski's definition of truth or not, we certainly must agree that any attempt to establish the truth value of a specific factual claim consists in reducing it through an appropriate formal analysis to some 'bottom' sentences which are empirically decidable. Under Tarski's conception, these bottom sentences are atomic sentences and all steps of the whole analysis through which the sentence in question becomes reduced to the atomic sentences is purely formal. Empirical procedures, and thus an experimenter or observer, his knowledge, his skill, become necessary.

Under the conception of the adequacy of theoretic laws I have outlined, not only the final step of the whole process of establishing the truth value, but also some intermediate steps may require the experimenter's competence; the reduction of theoretical laws to empirical claims is not merely a matter of formal analysis. Whether or not this means that the realist position is endangered depends on whether the expert's decisions are primarily subjective, culturally biased, or are primarily objective

resulting from an analysis of the structure of the world.

6. A FEW MORE REMARKS

It still may not be clear what kind of scientific realism I am ready to advocate and I do not think I have provided enough cues to answer to this question. To what I have said, something essential should be added.

Science is often viewed as a kind of linguistic activity. Its primary task is portrayed as producing hypothesis, theorems, theories, heuristic ideas, research programmes, etc. The set-theoretic approach *(non-statement view)* propounded by P. Suppes (1967) was a radical departure from this linguistic perspective. On the other hand, both the work done by Suppes himself and those who continued his ideas, notably van Fraassen (1980), F. Suppe (1989) and the large group of structuralists headed by J. Sneed and W. Stegmüller[24] leave no doubt that the model-theoretic approach can be interpreted as a switch from linguistic representation of the phenomena to their set-theoretic representation. Whether the latter is reducible to the former is debatable,[25] but anyway we remain in the world of signs, not things. In order to understand the phenomenon of science properly we must move out of it.

Scientific discoveries, thus science itself, have two levels. One of them consists of the phenomena scientists have learned to identify, even though they are not able to do this in an ideal, i.e. fully precise and fully reliable way. The other consists of representations of discovered phenomena. These are often inaccurate, sometimes misguided. One who wants to understand the growth of science must see this process both as change and as continuity. Ideas, concepts, theories, any linguistic representations, in fact any representations of the phenomena whatsoever undergo continuous changes. This part of science seems to be overwhelmed by the process of continuous modifications. So let us look at the other.

To separate phenomena from artifacts and even sheer illusions may not be easy.[26] But once a phenomenon is discovered it becomes a durable element of the scientific landscape, something which is there and, thus, something for which science must account. These are not empirical data, these are phenomena which scientific theories must save.[27] It may not be easy to render this metaphorically expressed idea in a sufficiently precise way. But it is precisely this idea which must be *clarified* if the realist philosophy of science is to be sustained.

Note that I have used the word 'clarify', not 'proved' or 'justified'. Philosophical ideas are too abstract and too remote from those parts of our knowledge that have direct empirical support to believe that they can be proved by a meticulously formed chain of derivations that starts from obvious, commonly agreed statements. In fact, the same can be said about many scientific ideas.

By clarifying an idea I mean rendering it in the form which makes explicit the relation it bears to other beliefs. It has to be shown that by appealing to the notion of a phenomenon one is able to provide a systematic and coherent account of science, scientific activity, outcomes of this activity, its peculiarities as well as its regularities. Thus, it has to be shown that the notion of a phenomenon, as 'a durable element of the scientific landscape', to use once more this metaphor, can be incorporated into a consistent system of beliefs on what science is. The philosophy of science thus formed must be continuous with science. This is a prerequisite of any realistic philosophy of science.

Am I not contradicting myself when I describe the aim of philosophizing as search for consistency rather than truth? No, because the hidden assumption of my line of thought is that the axiom of continuity with science guarantees that philosophy of science will not degenerate into a conglomerate, consistent perhaps but entirely arbitrary, and unrelated to one another systems of thought. I am far from presupposing in advance which kind of philosophy science will eventually prevail, what is the truth which will emerge from the debate. As the panorama of philosophy of science looks today, there are may deep, penetrating and, at the same time, conflicting ideas in this discipline. And curiously enough, even those who reject the principle of continuity of philosophy of science with science, and even those who go still further and question the very idea of the method of science, make considerable effort to argue their views following exactly the same formal rules that are observed in scientific reasoning. Surely, not everything goes in this discussion and not every point of view is equally good.

If my understanding of R. Rorty's appeal for demolishing epistemology and installing hermeneutics in its place (1979) is correct, his main concern is unlimited freedom of philosophical discussion. As he explains:

> Epistemology sees the hope of agreement as a token of the existence of common ground which, perhaps unbeknown to the speakers, unites them in a common rationality. For hermeneutics to be rational is to be willing to refrain from epistemology – from thinking that there is a special set

of terms in which all contributions to the conversation should be put – and to be willing to pick up the jargon of the interlocutor rather than translating it into one's own. ... For epistemology, conversation is implicit inquiry. For hermeneutics, inquiry is routine conversation.

I must confess that I am reluctant to participate in a conversation whose participants feel free to challenge any rule and any standpoint whatsoever. And the excuse that I am able to offer is expected. I do not believe that such a conversation has a reasonable chance of ending in something fruitful.

Unlimited freedom of discussion is what may be preached but not practised. Surely, it is not practised in science, and surely it is not practised in any quarter of philosophy. All of us who are engaged in one or another kind of intellectual activity find it tremendously difficult to cope with those challenges to our views which we consider reasonable. Overburdened, we narrow the area of our interest and focus our attention on selected standpoints. This might seem to be irrational, for we certainly may lose sight of something important. But the time and effort invested in exchange of ideas for which no common ground can be found and which are not united by some shared standards of rationality have little chance of paying. Anyway, to conclude the series of shameless commonplaces I have volunteered, scientific realism is an epistemological, not a hermeneutical doctrine.

The tendency to take a realist position, thus to believe that phenomena are part of the world, that they are governed by laws which again are objective, which we do not invent but discover, is more than a sociological phenomenon. For the scientist the hypothesis of the adequate accessibility of the world is part of his (or her) heuristic, one of the general guidelines that govern his efforts to solve the enigma of the mechanism of the world. It does not matter if he believes in the accessibility of the world or not. What matters is that he acts as if the world were adequately accessible – thus as if there was only one truth we may finally arrive at. No scientist advocates at the same time two opposite 'truths'. There is no room for doubletalk in science.

Actually, there is no room for doubletalk in any respectable philosophy. This why the relativist rarely, if ever, draws the conclusion from his own philosophy and admits that the principles he advocates are no better than those of his opponent.

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NOTES

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[1] See J. Margolis (1986) and also S. Haack (1987).

[2] J. Margolis (1986) calls the view characterized by these three assumptions *minimal realism.*

[3] For a brief but instructive discussion of the idea of incommensurability, see N. Rescher (1978).

[4] See S. Kuhn (1970. Another outstanding representative of the doctrine is P. Feyerabend (1975).

[5] Ajdukiewicz's conventionalism and related views have been surveyed by J. Giedymin (1982).

[6] See e.g. *Postscript* to the second edition of *The Structure of Scientific Revolutions.*

[7] In his work (1989), Putnam suggests that his *internal realism* can be more adequately referred to as *pragmatical realism.*

[8] This remark requires various qualifications. For an account of the views of both philosophers in the here discussed respect see S. Haack (1977).

[9] The conception of approximate truth discussed in Wójcicki (1979) is an exposition of the ideas presented earlier in Wójcicki (1969). A similar approach was developed by M. L. Dalla Chiara and G. Toraldo di Francia (1973).

[10] The earliest exposition of the conception was given in Tarski (1933). For an English version of this paper, or rather its improved German version, see Tarski (1956).

[11] See the next section.

[12] One of the earliest criticisms of Tarski's conception along this line is to be found in M. Black (1949).

[13] See M. Dummett (1978).

[14] The following quotation is from Margolis (1986), p. 111: "Relativism is an empirically motivated thesis to the effect that, in a particular sector of inquiry, it is methodologically advisable to retreat from insisting on a strong bipolar model of truth and falsity, while not denying that the affected propositions or claims *are* genuinely such, and as such, are to be ascribed suitable truth-like values – just such, in fact, but on the bipolar model (but not longer) would yield and confirm incompatible. In the spirit of the foregoing remarks, relativism is not only opposed to realism, but its advocates are positively committed to realism every bit as pertinaciously as are partisans of what – in a deliberately shadowy manner – I have called the 'standard' view. The trouble of course rests with what may be meant by speaking of *realism.*"

[15] I am using the term *phenomenon* in the most general sense. Roughly speaking, a phenomenon is any state of affairs, any event, any regularity, anything whatsoever which can be identified by any means accessible to an observer. Understood in this way the notion of phenomenon requires a rather scrupulous analysis, which certainly would go beyond the intended scope of this paper.

[16] The holistic character of meaning was, among others, emphasized by W. V. Quine in many of his writings. See e.g. (1961).

[17] These two examples were given by Dretske (1981).

[18] The reader who is interested in this analysis is advised to consult R. Montague (1962).

[19] See W. Balzer, C. U. Moulines, and J. Sneed (1987).

[20] See Section 2.

[21] This point, and more generally the view that the new theories do not falsify old ones but restrict the scope of their applicability was argued by F. Rohrlich (1988).

[22] See A. Grünbaum (1973).
[23] One of the questions widely discussed in philosophy of science is whether the way in which a superseded theory is related to the theory that supersedes it can be adequately described in a formal manner. The work done along this line, see e.g., recent publication os E. Scheibe (1986), (1987) is undoubtedly of great theoretic importance. How it is related to the one suggested here must be discussed on another occasion.
[24] The structuralist approach is surveyed in W. Balzer, C. U. Moulines, and J Sneed (1987).
[25] See D. Pearce (1987).
[26] Only recently have philosophers of science begun focusing their attention on this part of scientific activity. See, especially, M. Galison (1984) and J. Woodward (1989).
[27] For exactly the same point, see J. Woodward (1989).

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